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THE OPEN UNIVERSITY

Mathematics: A Third Level Course

PART V SHAPE OPERATORS

DIFFERENTIAL GEOMETRY

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

DIFFERENTIAL GEOMETRY PART V

SHAPE OPERATORS

A commentary on Chapter V of O'Neill's
Elementary Differential Geometry
Prepared by the Course Team

THE OPEN UNIVERSITY PRESS

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Further information on Open University courses may be obtained from The Admissions Office, The Open University, P.O. Box 48, Milton Keynes, MK7 6AB.

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Set Book

Barrett O'Neill: *Elementary Differential Geometry* (Academic Press, 1966). It is essential to have this book: the course is based on it and will not make sense without it.

Bibliography

The set books for M201, M231 and MST 282 are referred to occasionally; they are useful but not essential. They are:

D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins: *An Introduction to Linear Analysis* (Addison-Wesley, 1966).

E.D. Nering: *Linear Algebra and Matrix Theory* (John Wiley, 1970).

M. Spivak: *Calculus,* paperback edition (W. A. Benjamin/Addison-Wesley, 1973).

R.C. Smith and P. Smith: *Mechanics,* SI edition (John Wiley, 1972).

Conventions

Before starting work on this text, please read M334 Part Zero. Consult the Errata List and the Stop Press and make any necessary alterations for this chapter in the set book.

Unreferenced pages and sections denote the set book. Otherwise

O'Neill	denotes the set book;
Text	denotes the correspondence text;
KKOP	denotes *An Introduction to Linear Analysis* by D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins;
Nering	denotes *Linear Algebra and Matrix Theory by* E.D. Nering;
Spivak	denotes *Calculus* by M. Spivak;
Smith	denotes *Mechanics* by R.C. Smith and P. Smith.

Reference to Open University Courses in Mathematics take the form:

Unit M100 22, Linear Algebra I

Unit MST 281 10, Taylor Approximation

Unit M201 16, Euclidean Spaces I: Inner Products

Unit M231 2, Functions and Graphs

Unit MST 282 1, Some Basic Tools.

V.1 INTRODUCTION AND THE SHAPE OPERATOR OF $M \subset E^3$

Introduction

This section depends upon a knowledge of *curves in a surface* (Section IV.3) and *covariant derivatives* (Section II.5). We need no concept from Chapter IV from any sections later than Section IV.3 except that of *orientability* (Section IV.7) and that only so that we may use "orientable surface" as a shorthand for "surface which has a unit normal vector field defined on it". We need the fact that all surfaces are locally orientable; that is, given any point p in a surface M there is some neighbourhood N of p in M which is orientable. This is because there is some neighbourhood of p which lies in the image of a patch x, and this is oriented by the unit normal vector field $U = (x_u \times x_v)/\|x_u \times x_v\|$.

Throughout this chapter we confine our attention to surfaces M which are *connected* in the sense of O'Neill, Section IV.7; that is, given any points p, q in M there is a curve $\alpha\colon I \longrightarrow M$, where $[0, 1] \subset I$, such that $\alpha(0) = p$, $\alpha(1) = q$. This definition of "connected" is in general different from that given in *Unit M202 27, Connectedness* and *Unit M332 2, Continuous Functions* (in which M is connected if and only if it cannot be expressed as the disjoint union of two non-empty open subsets of M), but it may be shown that for surfaces in E^3 the two definitions are equivalent (where, for the M202 definition, the topology on M is that of a subspace of R^3 with the usual topology). Thus if you have done M202 or M332 you may use whichever definition of "connected" that you find easier.

In Chapter IV we established techniques for describing and investigating surfaces in E^3. Now we can use these techniques to look at some geometric properties of surfaces in E^3. One such property is the *shape* of a surface: we can describe the shape of a surface M near a point p in M by measuring the rate of change of a unit normal vector field to M along curves in M through p. We do this by using the covariant derivative. We shall need to modify somewhat our previous definition of covariant derivative so that we can apply it to vector fields defined on surfaces only. In Section II.5 $\nabla_v Z$ was defined as

(i) $\nabla_v Z = Z(p + tv)'(0)$:

in Exercise II.5.6 and Lemma II.5.2 it was shown that this definition is equivalent to

(ii) $\nabla_v Z = (Z \circ \alpha)'(0)$ for any curve α with $\alpha'(0) = v$

and

(iii) $\nabla_v Z = \Sigma v[z_i]\, U_i(p)$ where z_i are the coordinate functions of Z.

If we restrict Z to be a vector field defined on a surface M, definition (i) is no longer available to us, as the curve $t \longmapsto p + tv$ may not lie within M. However, if we also restrict v to be a tangent vector to E^3 which is tangent to M then by definition there is some curve α in M with $\alpha'(0) = v$: now everything in definition (ii) is meaningful, and accordingly we take the following as our new definition.

Definition If Z is a vector field on a surface M and v is a vector tangent to M, then the covariant derivative $\nabla_v Z$ is the common value of $(Z \circ \alpha)'(0)$ for all curves α in M with $\alpha'(0) = v$.

As before, this is equivalent to

$$\begin{aligned}\nabla_v Z &= \Sigma v[z_i]\, U_i(p)\\ &= \Sigma (z_i \circ \alpha)'(0)\, U_i(p)\end{aligned}$$

for all curves α in M with $\alpha'(0) = v$ (by Definition IV.3.10), and the arguments in

Comment (v) to Section IV.3 (Text, page 35) which show that $v[z_i]$ is independent of the choice of α can be extended to show that $\nabla_v Z$ is also independent of the choice of α. Moreover, if Z is the restriction to M of a vector field W on E^3 (equivalently, if z_i is the restriction to M of a differentiable function w_i on E^3 for $i = 1, 2, 3$) then the covariant derivative $\nabla_v Z$ obtained from the above definition is the same as the covariant derivative $\nabla_v W$ obtained as in Section II.5: thus we may evaluate the covariant derivative by any method from Section II.5 in such cases.

We spend the first three sections of this chapter defining, and giving geometric meaning to, concepts such as *shape, normal curvature, Gaussian curvature,* and defer until Section V.4 explicit methods for calculating these for general surfaces.

READ: Introduction to Chapter V and Section V.1 (pages 189-193).

Comments

(i) *Page 190: lines 14-16* We have not proved that there are exactly two unit normal vector fields on a connected orientable surface, as the proof requires more work on connectedness outside the main stream of this course. However, the idea of the proof is very simple: if Z is a unit normal vector field then at each point Z must be either U or −U, and on a connected surface Z cannot suddenly flip over from U to −U without losing its differentiability.

(ii) *Page 192: line 1* The vector field used here and in Example IV.3.9, $X = \Sigma x_i U_i$, is an interesting one that will be used later in the chapter. It has the property that

$$\nabla_v X = \Sigma v[x_i]\; U_i(p) = \Sigma v_i U_i(p) = v$$

and similarly,

$$\nabla_V X = V.$$

(iii) *Page 193: the last paragraph* The name "symmetric linear transformation" derives from the fact that the matrix of a symmetric linear transformation with respect to any orthonormal basis is a symmetric matrix. In the two-dimensional case this means that with respect to an orthonormal basis the matrix has the form

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where $a_{12} = a_{21}$. The significance of such transformations is that they are always diagonalizable; that is, there is some orthonormal basis with respect to which the matrix is diagonal. Such a basis consists of eigenvectors of the linear transformation. For example, in the two-dimensional case the matrix with respect to an eigenvector basis has the form

$$B = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

where k_1 and k_2 are eigenvalues of the transformation. The eigenvalues k_1 and k_2 may possibly be equal, and either of them may be zero. O'Neill uses the words *characteristic vector, characteristic value* to refer to eigenvectors and eigenvalues.

O'Neill also refers to the *determinant* and *trace* of a linear transformation. You have met determinants already in this course. The determinant of a linear transformation T from a finite-dimensional vector space V into itself is defined to be the determinant of any matrix which represents the transformation T; it can be shown that this is independent of the choice of basis. In the above examples

$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

and

$$\det B = k_1k_2.$$

The trace of a matrix is defined to be the sum of its diagonal terms. Thus

$$\text{trace } A = a_{11} + a_{22}$$

and

$$\text{trace } B = k_1 + k_2.$$

The trace of a linear transformation T from a finite-dimensional vector space V into itself is defined to be the trace of the matrix of T with respect to any basis of V: again it can be shown that this is independent of the choice of basis. For example, the trace of the identity mapping from E^n to E^n is simply n.

Additional Text

We shall need the result of the following exercise throughout the remainder of Chapter V.

1. Page 194, Exercise 1. (HINT: Use the result of Exercise II.5.6(a).)

Summary

Notation

$\nabla_v Z$	Page 189, line -6
S_p	Page 190, Definition 1.1
S	Page 191, line -8
$f_x, f_y, f_{xx}, f_{xy}, f_{yy}$	Text: page 8, Exercise 4

Definitions

(i)	The covariant derivative $\nabla_v Z$ on a surface	Text, page 5
(ii)	The shape operator at p, S_p	Page 190, Definition 1.1
(iii)	The shape operator S	Page 191, line -9
(iv)	The trace of a linear operator	Text, page 7

Examples

(i)	If M is a sphere of radius r, then $S = -\frac{1}{r} I.$	Pages 191-192, Example 1.3(1)
(ii)	If M is a circular cylinder of radius r, then $S(v) = 0$ if v is tangent to a ruling of M and $S(v) = -v/r$ if v is tangent to a cross-sectional circle of M.	Page 192, Example 1.3(3)

Results

(i)	S_p is a symmetric linear transformation from $T_p(M)$ to $T_p(M)$.	Page 191, Lemma 1.2 and Page 193, Lemma 1.4
(ii)	$S(\alpha') = -U'$, where U is restricted to α.	Page 194, Exercise 1
(iii)	If $X = \Sigma x_i U_i$, then $\nabla_v X = v$.	Text, page 6

Techniques

(i)	Calculating the covariant derivative of a vector field on a surface using	
	(a) $\nabla_v Z = (Z \circ \alpha)'(0)$ $\quad$ for $\alpha'(0) = v_p$.	Page 189, Method 1
	(b) $\nabla_v Z = \Sigma (z_i \circ \alpha)'(0) U_i(p)$ $\quad$ for $\alpha'(0) = v_p$.	Page 190, Method 2
(ii)	Where Z, z_i are restrictions to M of a vector field on E^3 and differentiable functions on E^3, calculating the covariant derivative using	
	(a) $\nabla_v Z = \Sigma z_i(p + tv)'(0)\, U_i(p)$	Page 78, Lemma II.5.2
	(b) $\nabla_v Z = \Sigma v[z_i]\, U_i(p)$, where $v[z_i]$ is evaluated using $U_i(p)[x_j] = \delta_{ij}$ and the linear and Leibnizian properties.	Page 80, Corollary II.5.4
(iii)	Finding the matrix representing S_p with respect to some basis for $T_p(M)$, and hence determining the rank of S_p (that is, the number of linearly independent rows) and its effect on the whole of $T_p(M)$.	Page 190, Definition 1.1

Exercises

Technique (i)

2. If $x : D \longrightarrow M$ is a patch, Z a vector field on M, and w is the tangent vector $x_u(u_0, v_0)$ to M at $x(u_0, v_0)$, show that

$$\nabla_w Z = \frac{\partial}{\partial u}(Z(x))(u_0, v_0) = \sum_i \frac{\partial z_i(x)}{\partial u}(u_0, v_0)\, U_i\,(x(u_0, v_0)).$$

Technique (ii)(a)

3. Calculate $\nabla_v Z$ for $v = (0, 1, 0)$ at the point $(r, 0, 0)$ on the cylinder $x^2 + y^2 = r^2$ when $Z = -yU_1 + xU_2$.

Technique (ii)(b)

4. Page 194, Exercise 2. What is the matrix of S_0 with respect to $\{u_1, u_2\}$?. We shall use the result of this exercise in the next section, but it is not worth remembering for its own sake. The symbol f_x denotes $\partial f/\partial x$, $f_y = \partial f/\partial y$, $f_{xx} = \partial^2 f/\partial x^2$, $f_{xy} = \partial^2 f/\partial x \partial y$, $f_{yy} = \partial^2 f/\partial y^2$.

Technique (iii)

5. Page 194, Exercise 3. Give the matrix of S_0 with respect to $\{u_1, u_2\}$ in each case.

Optional Section – The Gauss Mapping

Historically the unit normal vector field U was studied by investigating the Gauss mapping. Although this is outside the scope of this course, the Gauss mapping does play an important role in differential geometry: for example, it is used in the second of the sections in O'Neill on Integration (VI.7), which are omitted from M334 but which you may like to read later. If you would like to study the Gauss mapping, read the definition given in Exercise 4 on page 194 and then do the following exercises.

6. Page 194, Exercise 4.

7. Page 195, Exercise 5.

8. Page 195, Exercise 6.

9. Page 195, Exercise 7.

Solutions

1. Page 194, Exercise 1.

We need to show that

$$S(\alpha'(t_0)) = -(U(\alpha))'(t_0)$$

for all t_0 in the domain of α. Putting $v = \alpha'(t_0)$,

$$\begin{aligned} S(\alpha'(t_0)) &= S(v) \\ &= -\nabla_v U, && \text{by Definition V.1.1,} \\ &= -(U \circ \beta)'(0), && \text{by definition of the covariant derivative,} \end{aligned}$$

where β is any curve in M with $\beta'(0) = \alpha'(t_0)$. An obvious choice for β is

$$\beta : t \longmapsto \alpha(t + t_0).$$

Now

$$U(\beta(t)) = U(\alpha(t + t_0))$$

so

$$(U(\beta))'(t) = (U(\alpha))'(t + t_0)$$

so

$$(U(\beta))'(0) = (U(\alpha))'(t_0).$$

Thus

$$S(\alpha'(t_0)) = -(U(\beta))'(0) = -(U(\alpha))'(t_0).$$

Since we are considering just the restriction of U to the curve α, so that we are thinking of U as a vector field on a curve, we may write U' for $(U(\alpha))'$. Thus the last equation becomes, on omitting t_0,

$$S(\alpha') = -U'.$$

2. The vector $w = x_u(u_0, v_0)$ is the tangent vector to the curve $u \longmapsto x(u, v_0)$ at the point $x(u_0, v_0)$, so if we take α to be the curve

$$\alpha : t \longmapsto x(u_0 + t, v_0)$$

lying in M, then $x_u(u_0, v_0) = \alpha'(0)$. We use Method 1 on pages 189-190 to obtain

$$\begin{aligned}\nabla_w Z &= (Z(\alpha))'(0)\\ &= \frac{d}{dt}(Z(\alpha(t)))\Big|_{t=0}\\ &= \frac{d}{dt}(Z(x(u_0+t, v_0)))\Big|_{t=0}\\ &= \frac{\partial}{\partial u}(Z(x))(u_0, v_0) \qquad \text{by definition of partial differentiation.}\end{aligned}$$

Similarly, $(z_i(\alpha))'(0) = \frac{\partial}{\partial u}(z_i(x))(u_0, v_0)$, and so Method 2 gives

$$\nabla_w Z = \Sigma\,(z_i(\alpha))'(0)\; U_i\,(x(u_0, v_0)) = \Sigma \frac{\partial}{\partial u}(z_i(x))(u_0, v_0)\; U_i\,(x(u_0, v_0)).$$

3. $\nabla_v(-yU_1 + xU_2) = (-y(p + tv))'(0)\; U_1(p) + (x(p + tv))'(0)\; U_2(p).$

In this case

$$\begin{aligned}p + tv &= (r, 0, 0) + t(0, 1, 0)\\ &= (r, t, 0)\end{aligned}$$

so

$$-y(p + tv) = -t$$

so

$$(-y(p + tv))'(0) = -1,$$

and

$$x(p + tv) = r$$

so

$$(x(p + tv))'(0) = 0.$$

Thus

$$\nabla_v(-yU_1 + xU_2) = -U_1(p) = -U_1(r, 0, 0).$$

4. Page 194, Exercise 2.

(a) Since M is given implicitly as $M : g = 0$, where

$$g(x, y, z) = z - f(x, y),$$

we know from Lemma IV.3.8 that ∇g is a normal vector field on M.

$$\begin{aligned}\nabla g &= \frac{\partial g}{\partial x} U_1 + \frac{\partial g}{\partial y} U_2 + \frac{\partial g}{\partial z} U_3\\ &= -f_x U_1 - f_y U_2 + U_3.\end{aligned}$$

This has norm $(f_x{}^2 + f_y{}^2 + 1)^{\frac{1}{2}}$, and so a unit normal vector field to M is given by

$$U = \frac{-f_x U_1 - f_y U_2 + U_3}{(f_x{}^2 + f_y{}^2 + 1)^{\frac{1}{2}}}.$$

Now $U(0, 0, 0) = (0, 0, 1)$ because $f_x(0, 0) = f_y(0, 0) = 0$. Tangent vectors at $(0, 0, 0)$ are tangent to M if and only if they are orthogonal to $U(0, 0, 0) = (0, 0, 1)$. The tangent vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are both orthogonal to $(0, 0, 1)$, and so they are tangent to M at the origin.

(b) By Method 2 on page 190,

$$\nabla_{\mathbf{u}_1} U = \mathbf{u}_1\left[\frac{-f_x}{(f_x{}^2 + f_y{}^2 + 1)^{\frac{1}{2}}}\right] U_1(0) + \mathbf{u}_1\left[\frac{-f_y}{(f_x{}^2 + f_y{}^2 + 1)^{\frac{1}{2}}}\right] U_2(0)$$

$$+ \mathbf{u}_1\left[\frac{1}{(f_x{}^2 + f_y{}^2 + 1)^{\frac{1}{2}}}\right] U_3(0).$$

We know that S_p maps $T_p(M)$ to $T_p(M)$ and so the component of $\nabla_{\mathbf{u}_1} U$ in the $U_3(0)$ direction is zero.

Now $\mathbf{u}_1[h] = (\partial h/\partial x)(0, 0, 0)$ for any function h. Since we know that $f_x(0, 0) = f_y(0, 0) = 0$ we can ignore any terms having f_x or f_y as a factor when we have differentiated, so

$$\begin{aligned}\mathbf{u}_1\left[\frac{-f_x}{(f_x{}^2 + f_y{}^2 + 1)^{\frac{1}{2}}}\right] &= \frac{\partial}{\partial x}\left(\frac{-f_x}{(f_x{}^2 + f_y{}^2 + 1)^{\frac{1}{2}}}\right)(0, 0, 0)\\ &= \frac{-f_{xx}}{(f_x{}^2 + f_y{}^2 + 1)^{\frac{1}{2}}}(0, 0, 0)\\ &= -f_{xx}(0, 0);\end{aligned}$$

Similarly

$$\mathbf{u}_1\left[\frac{-f_y}{(f_x{}^2 + f_y{}^2 + 1)^{\frac{1}{2}}}\right] = -f_{xy}(0, 0)$$

Thus

$$\nabla_{\mathbf{u}_1} U = -f_{xx}(0, 0)\, U_1(0) - f_{xy}(0, 0)\, U_2(0).$$

Now

$$\begin{aligned}S(\mathbf{u}_1) &= -\nabla_{\mathbf{u}_1} U\\ &= f_{xx}(0, 0)\, U_1(0) + f_{xy}(0, 0)\, U_2(0)\\ &= f_{xx}(0, 0)\mathbf{u}_1 + f_{xy}(0, 0)\mathbf{u}_2.\end{aligned}$$

In an exactly similar manner, since $\mathbf{u}_2[h] = (\partial h/\partial y)(0, 0, 0)$, we can show that

$$S(\mathbf{u}_2) = f_{xy}(0, 0)\mathbf{u}_1 + f_{yy}(0, 0)\mathbf{u}_2.$$

The tangent vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ form a basis for $T_0(M)$. The two equations

$$S(\mathbf{u}_1) = f_{xx}(0, 0)\mathbf{u}_1 + f_{xy}(0, 0)\mathbf{u}_2$$

$$S(\mathbf{u}_2) = f_{xy}(0, 0)\mathbf{u}_1 + f_{yy}(0, 0)\mathbf{u}_2$$

tell us that the matrix of S with respect to this basis is

$$\begin{pmatrix} f_{xx}(0, 0) & f_{xy}(0, 0)\\ f_{xy}(0, 0) & f_{yy}(0, 0)\end{pmatrix}$$

5. Page 194, Exercise 3.

In each case we can put $z = f(x, y)$ and show that

$$f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0.$$

Then the result of the previous exercise tells us that, with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$, the matrix of $\mathbf{S_0}$ is

$$\begin{pmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{xy}(0,0) & f_{yy}(0,0) \end{pmatrix}$$

Expressing every vector in $T_0(M)$ in terms of the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ we see that $S(a\mathbf{u}_1 + b\mathbf{u}_2) = c\mathbf{u}_1 + d\mathbf{u}_2$, where c and d are obtained from the equation:

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{xy}(0,0) & f_{yy}(0,0) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

The rank of $\mathbf{S_0}$ can be simply read off as the number of linearly independent rows of the matrix.

(a) In this case $f(x, y) = xy$.

$$f_x(x, y) = y \qquad f_{xx}(x, y) = 0$$
$$f_y(x, y) = x \qquad f_{xy}(x, y) = 1$$
$$f_{yy}(x, y) = 0$$

The matrix of S_0 with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and so

$$S(a\mathbf{u}_1 + b\mathbf{u}_2) = b\mathbf{u}_2 + a\mathbf{u}_1$$

and S_0 has rank 2.

(b) Now $f(x, y) = 2x^2 + y^2$.

$$f_x(x, y) = 4x \qquad f_{xx}(x, y) = 4$$
$$f_y(x, y) = 2y \qquad f_{xy}(x, y) = 0$$
$$f_{yy}(x, y) = 2$$

The matrix of S_0 with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ is

$$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

and so

$$S(a\mathbf{u}_1 + b\mathbf{u}_2) = 4a\mathbf{u}_1 + 2b\mathbf{u}_2$$

and S_0 has rank 2.

(c) Now $f(x, y) = (x + y)^2$.

$$f_x(x, y) = 2(x + y) \qquad f_{xx}(x, y) = 2$$
$$f_y(x, y) = 2(x + y) \qquad f_{xy}(x, y) = 2$$
$$f_{yy}(x, y) = 2.$$

The matrix of S_0 with respect to the basis $\{u_1, u_2\}$ is

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

and so

$$S(au_1 + bu_2) = 2(a + b)(u_1 + u_2)$$

and S_0 has rank 1.

(d) Now $f(x, y) = xy^2$.

$$f_x(x, y) = y^2 \qquad f_{xx}(x, y) = 0$$
$$f_y(x, y) = 2xy \qquad f_{xy}(x, y) = 2y$$
$$f_{yy}(x, y) = 2x$$

The matrix of S_0 with respect to the basis $\{u_1, u_2\}$ is

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

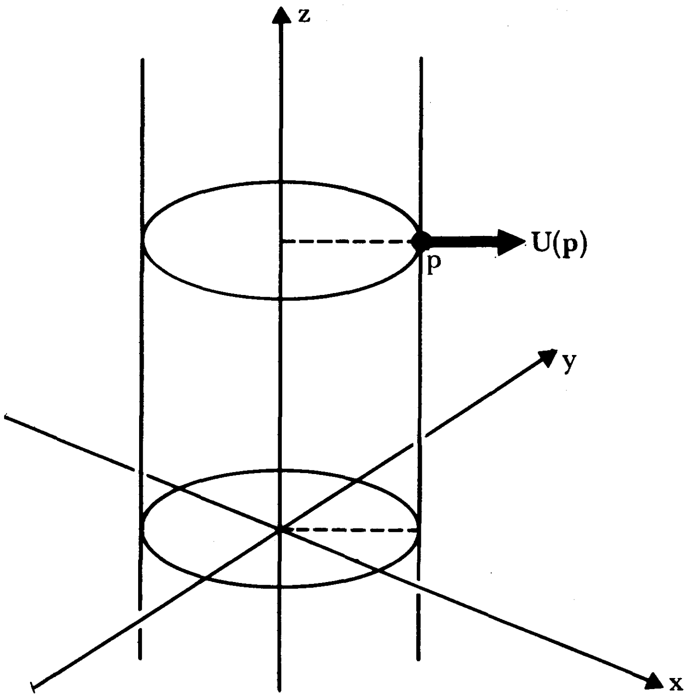
and so

$$S(au_1 + bu_2) = 0$$

and S_0 has rank 0.

6. Page 194, Exercise 4.

(a) The "outwards" unit normal on the cylinder $x^2 + y^2 = r^2$ at the point (p_1, p_2, p_3) is $(p_1, p_2, 0)/r$.



Thus G(M) is the set of all points $(p_1, p_2, 0)/r$ with $p_1^2 + p_2^2 = r^2$; that is, G(M) is the unit circle in the xy plane.

(b) The cone M: $g = (x^2+y^2)^{\frac{1}{2}} - z = 0$ is a "single" cone with positive z coordinates. We must assume that the vertex 0 is omitted, or else M would not be a surface. As a normal vector field on M we can take

$$\begin{aligned}\nabla g &= \frac{x}{(x^2+y^2)^{\frac{1}{2}}} U_1 + \frac{y}{(x^2+y^2)^{\frac{1}{2}}} U_2 - U_3\\ &= \frac{x}{z} U_1 + \frac{y}{z} U_2 - U_3.\end{aligned}$$

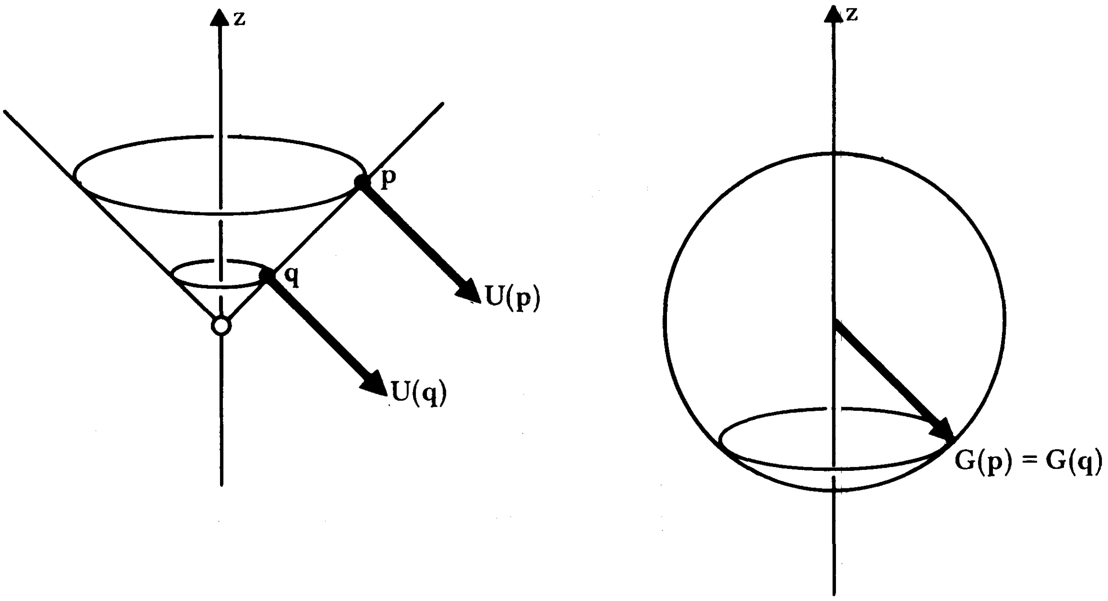
Then

$$\begin{aligned}\|\nabla g\| &= \left(\frac{x^2+y^2}{z^2} + 1\right)^{\frac{1}{2}}\\ &= \sqrt{2}\end{aligned}$$

because $x^2 + y^2 = z^2$, and so a unit normal vector field on M is

$$\frac{1}{\sqrt{2}z}(xU_1 + yU_2 - zU_3).$$

Thus G(M) is the set of all points $(p_1, p_2, -p_3)/\sqrt{2}p_3$ with $p_3 = (p_1{}^2 + p_2{}^2)^{\frac{1}{2}}$, which is the circle which is formed by the intersection of the unit sphere Σ with the plane $z = -1/\sqrt{2}$.



(Of course, we could have chosen the opposite unit normal, in which case G(M) would have been the intersection of Σ with the plane $z = 1/\sqrt{2}$.)

(c) As noted in Example V.1.3(2), the unit normal vector field to a plane is parallel; that is, its Euclidean coordinate functions are all constants. Hence if M is a plane then G(M) is a single point. In this case M : g = 0, where $g = x + y + z$:

$$\nabla g = U_1 + U_2 + U_3$$

and so G(M) is the single point $(1, 1, 1)/\sqrt{3}$.

(d) Using the gradient, we find that

$$(x - 1)U_1 + yU_2 + (z + 2)U_3$$

is a unit normal vector field on M: $(x - 1)^2 + y^2 + (z + 2)^2 = 1$.

Thus

$$\begin{aligned} G(M) &= \{(p_1 - 1, p_2, p_3 + 2): (p_1 - 1)^2 + p_2^2 + (p_3 + 2)^2 = 1\} \\ &= \{(q_1, q_2, q_3): q_1^2 + q_2^2 + q_3^2 = 1\} \\ &= \Sigma. \end{aligned}$$

In other words, in this case G(M) is the whole of the unit sphere.

7. Page 195, Exercise 5.

We use the parametrization

$$x(u, v) = ((R + r \cos u)\cos v, (R + r \cos u)\sin v, r \sin u)$$

of the torus T. We know that $x_u \times x_v$ is a normal vector field on T. Now

$$x_u(u, v) = (-r \sin u \ \cos v, \ -r \sin u \ \sin v, \ r \cos u)$$

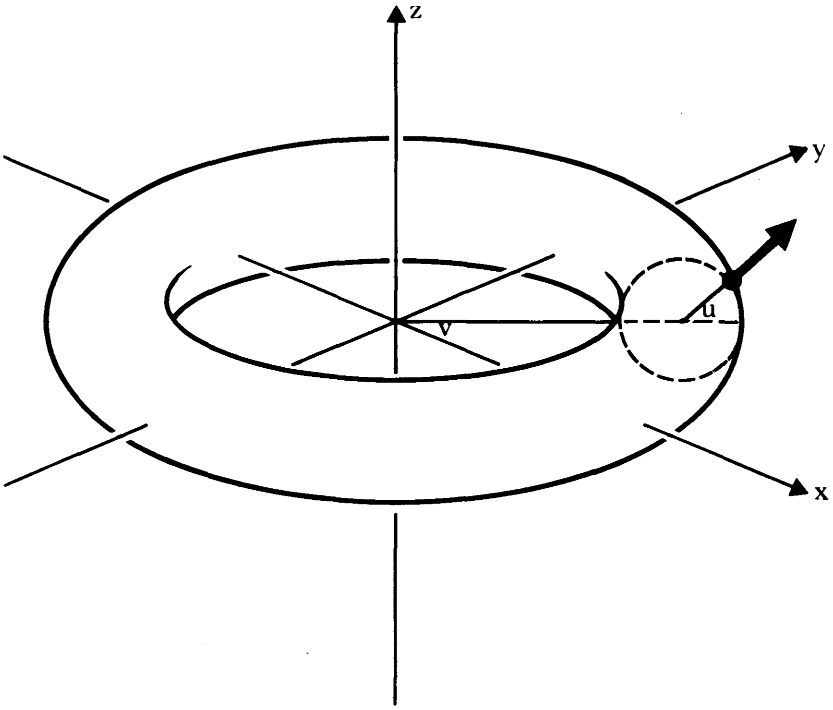
$$x_v(u, v) = (-(R + r \cos u)\sin v, \ (R + r \cos u)\cos v, \ 0)$$

and so

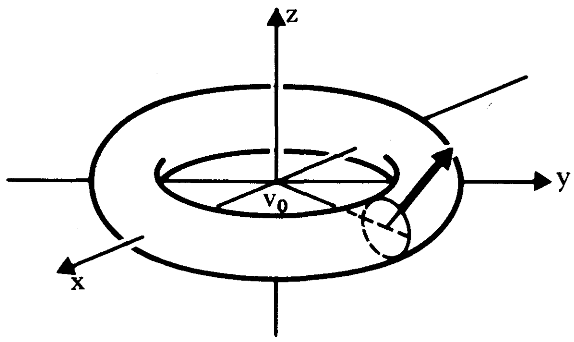
$$(x_u \times x_v)(u, v) = -r(R + r \cos u)(\cos u \ \cos v, \ \cos u \ \sin v, \ \sin u).$$

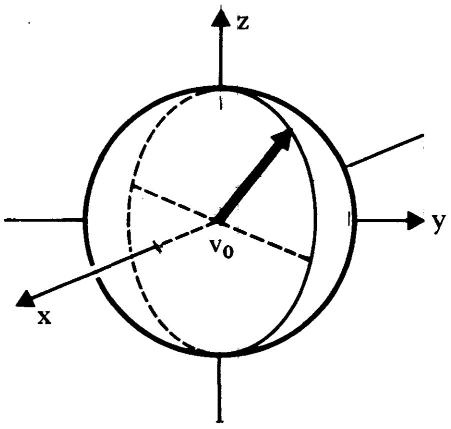
Thus the outward unit normal at x(u, v) is given by

$$(\cos u \ \cos v, \ \cos u \ \sin v, \ \sin u).$$

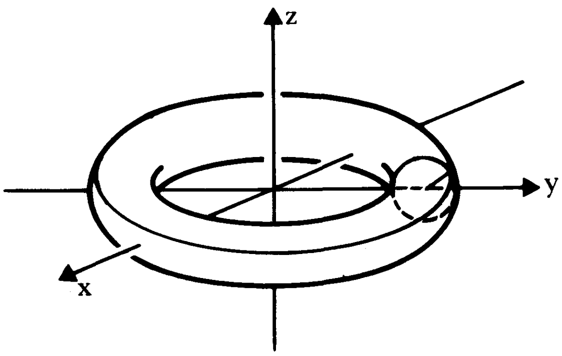


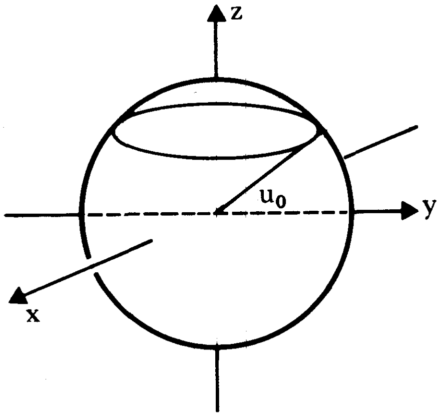
The coordinates of this are identical to those of the geographical parametrization of the unit sphere, except that u and v are interchanged and may each take any real value. A typical meridian of T is the u-parameter curve $v = v_0$: G maps this to the vertical great circle on Σ which is the union of the meridian $u = v_0$ with the meridian $u = v_0 + \pi$ and the North and South poles.

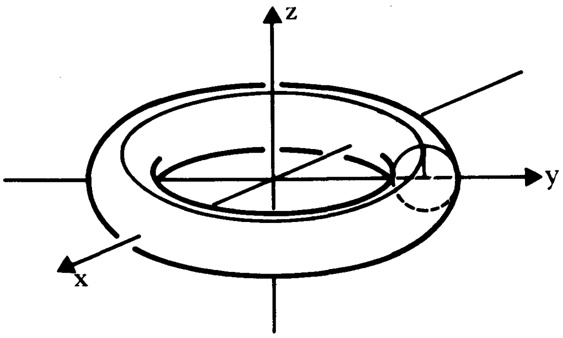


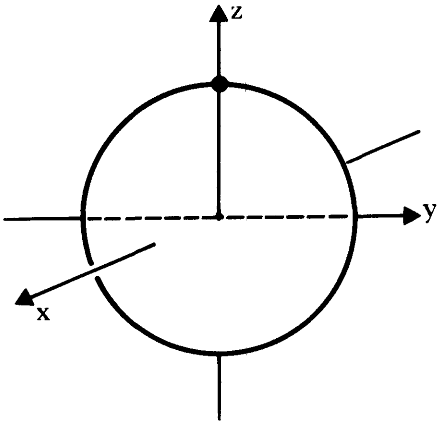


A typical parallel of T is the v-parameter curve $u = u_0$: G maps this to the parallel on Σ given by $v = u_0$ when u_0 is not of the form $n\pi + \pi/2$. These particular values of u_0 correspond to the top circle on T, which is mapped to the North pole on Σ, and the bottom circle on T, which is mapped to the South pole on Σ.

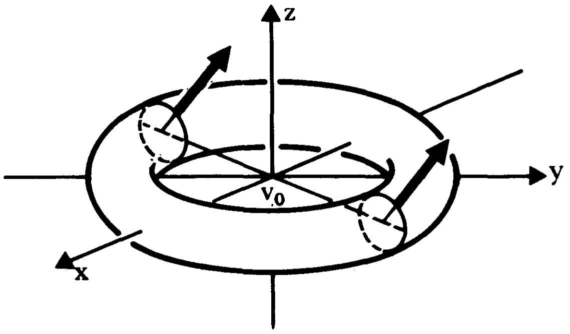


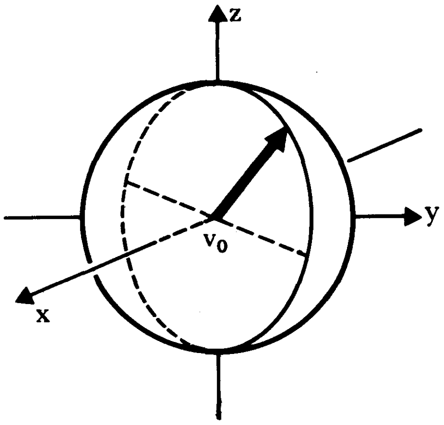






The meridian $v = v_0$ on T has an "opposite" meridian $v = v_0 + \pi$ (the meridian $v = v_0 + 2\pi$ is of course the same as the meridian $v = v_0$). G maps each of these meridians to the whole of the same vertical great circle on Σ.





We have already seen that the North and South poles of Σ are each the image of a whole circle on T. Every other point of Σ lies on just one vertical great circle and so is the image of exactly two points of T, which lie on opposite meridians of T.

8. Page 195, Exercise 6.

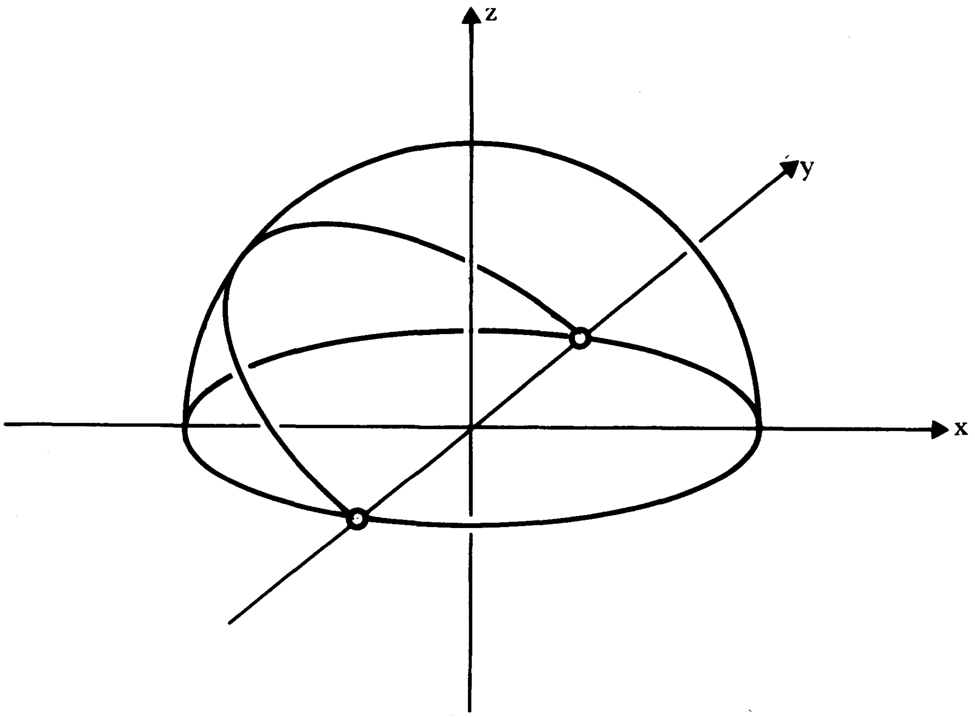
The saddle surface is $M : z = f(x, y)$, where $f(x, y) = xy$. The unit normal vector field U found in Exercise V.1.2 is

$$\frac{-f_xU_1 - f_yU_2 + U_3}{(1 + f_x{}^2 + f_y{}^2)^{\frac{1}{2}}}$$
$$= \frac{-yU_1 - xU_2 + U_3}{(1 + y^2 + x^2)^{\frac{1}{2}}}.$$

The line $y = c$ on M is mapped by G to the set of points

$$\left\{\frac{(-c, -p_1, 1)}{(1 + c^2 + p_1{}^2)^{\frac{1}{2}}} : p_1 \in R\right\}.$$

The point q in Σ belongs to this set if and only if q_3 is positive and $q_1 = -cq_3$. Thus G maps the line $y = c$ to the semicircle which is the intersection of Σ, the plane $x = -cz$ and the set $\{p \in E^3 : z(p) > 0\}$.



Each point of M lies on one of the lines $y = c$ for some real value of c. As c varies over R the images of these lines cover the whole of the northern hemisphere, $\{p \in \Sigma : z(p) > 0\}$.

9. Page 195, Exercise 7.

Let v be a vector tangent to M. Then there is a curve α in M with $\alpha'(0) = v$. Then

$$\begin{aligned} S(v) &= S(\alpha'(0)) \\ &= -(U(\alpha))'(0), \qquad \text{by Exercise V.1.1.} \end{aligned}$$

Now

$$\begin{aligned} G_*(v) &= G_*(\alpha'(0)) \\ &= (G(\alpha))'(0), \qquad \text{by Definition IV.5.3.} \end{aligned}$$

Since G and U have the same Euclidean coordinate functions, $U(\alpha)'(0)$ and $G(\alpha)'(0)$ also have the same coordinates; that is, the vectors $-S(v)$ and $G_*(v)$ are parallel.

V.2 NORMAL CURVATURE

Introduction

This section follows on directly from Section V.1. We also need to use the concepts of *eigenvector* and *eigenvalue* (see *Unit M201 5, Determinants and Eigenvalues*) and a theorem from *Unit M201 24, Orthogonal and Symmetric Transformations.*

In the previous section we saw that S_p is a linear transformation of $T_p(M)$ to itself, and accepted (leaving the proof until Section V.4) that S_p is a symmetric linear transformation; that is,

$$S_p(v)\cdot w = v\cdot S_p(w).$$

You have met symmetric linear transformations before, in *Unit M201 24, Orthogonal and Symmetric Transformations,* where the following important theorem was proved (page 37): if S is a symmetric linear transformation of a finite dimensional vector space V to itself then V has an orthonormal basis consisting of eigenvectors of S. Applying this theorem to S_p, we see that

$$T_p(M) \text{ has a basis } e_1, e_2 \text{ where } \|e_1\| = \|e_2\| = 1,\ e_1\cdot e_2 = 0,$$

and

$$S_p(e_1) = k_1e_1, \quad S_p(e_2) = k_2e_2,$$

where k_1, k_2 are the eigenvalues of S_p corresponding to e_1, e_2. The tangent vectors e_1, e_2 are called *principal vectors* at p, and k_1, k_2 are called *principal curvatures* at p. That is, we make the following definitions.

Definition. The principal curvatures at p are the eigenvalues of S_p.

Definition. The principal vectors at p are the unit eigenvectors of S_p.

Definition. A principal direction at p is the one-dimensional subspace of $T_p(M)$ spanned by a principal vector at p.

Thus, if v is tangent to M at p, $S_p(v)$ is a multiple of v if and only if v is an eigenvector of S_p, which happens if and only if v lies in a principal direction at p. Notice that, with this definition, a principal direction is not oriented; if v is an eigenvector of S_p then v and $-v$ both belong to the same principal direction.

With respect to the basis $\{e_1, e_2\}$ the linear transformation S_p has matrix

$$\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

In the particular case when $k_1 = k_2 = k$ this becomes kI, and so S_p is simply multiplication by the scalar k; that is, every vector of $T_p(M)$ is an eigenvector of S_p with eigenvalue k. In this case we call p *umbilic.*

Definition p is umbilic if S_p is scalar multiplication.

If v is a nonzero eigenvector of S_p with eigenvalue k_i then

$$S_p(v)\cdot v = (k_iv)\cdot v = k_i(v\cdot v)$$

and so

$$k_i = \frac{S_p(v)\cdot v}{v\cdot v}.$$

In particular, since e_1 and e_2 are unit vectors, the eigenvalues of S_p are given by

$$k_i = S_p(e_i)\cdot e_i.$$

In fact we have the following theorem.

Theorem V.2.A The principal curvatures at p are the minimum and maximum values of $S_p(\mathbf{u})\cdot\mathbf{u}$ for unit tangent vectors $\mathbf{u}$ in $T_p(M)$. Moreover these minimum and maximum values occur only when $\mathbf{u}$ is a principal vector at p. In particular, $S_p(\mathbf{u})\cdot\mathbf{u}$ is constant if and only if p is umbilic.

Proof Let $\{e_1, e_2\}$ be an orthonormal eigenvector basis of $T_p(M)$. If $\mathbf{u}$ is any unit tangent vector in $T_p(M)$, we can write

$$\mathbf{u} = \cos\vartheta\ e_1 + \sin\vartheta\ e_2$$

for some angle ϑ. Since $S_p(\mathbf{u})$ is a linear transformation

$$\begin{aligned} S_p(\mathbf{u}) &= \cos\vartheta\ S_p(e_1) + \sin\vartheta\ S_p(e_2) \\ &= k_1 \cos\vartheta\, e_1 + k_2 \sin\vartheta\, e_2 \end{aligned}$$

and so

$$\begin{aligned} S_p(\mathbf{u})\cdot\mathbf{u} &= (k_1 \cos\vartheta\, e_1 + k_2 \sin\vartheta\, e_2)\cdot(\cos\vartheta\, e_1 + \sin\vartheta\, e_2) \\ &= k_1 \cos^2\vartheta + k_2 \sin^2\vartheta \\ &= k_2 + (k_1 - k_2)\cos^2\vartheta. \end{aligned}$$

As $\cos^2\vartheta$ varies between 1 and 0, $S_p(\mathbf{u})\cdot\mathbf{u}$ varies between k_1 and k_2, which are therefore the minimum and maximum values (not necessarily in that order) of $S_p(\mathbf{u})\cdot\mathbf{u}$ for unit vectors $\mathbf{u}$ in $T_p(M)$. Moreover if $k_1 \neq k_2$, then these extreme values occur only when $\cos^2\vartheta = 0$ or 1, that is when $\mathbf{u} = \pm e_1$ or $\pm e_2$.

In the case when p is umbilic, with $k_1 = k_2 = k$, $S_p(\mathbf{u}) = k\mathbf{u}$ for all unit vectors $\mathbf{u}$ in $T_p(M)$, and so $S_p(\mathbf{u})\cdot\mathbf{u} = k$ is constant. Conversely, if $S_p(\mathbf{u})\cdot\mathbf{u}$ is constant then $k_1 = k_2$ and so p is umbilic.

O'Neill approaches in a more geometric way the concepts that we have just defined algebraically. The geometry also naturally leads us to consider the importance of the principal vectors and curvatures, and the *normal curvature function*

$$\mathbf{u} \longmapsto S_p(\mathbf{u})\cdot\mathbf{u} \qquad \mathbf{u} \in T_p(M), \|\mathbf{u}\| = 1.$$

We have kept the notation S_p in this introduction to emphasize that S_p is a linear transformation of $T_p(M)$. However, in general it is quite safe to abbreviate S_p to S, as O'Neill has already started doing, for if v is a tangent vector tangent to M at p then S(v) must mean $S_p(v)$.

READ: Section V.2 (pages 195-202), omitting the proof of Theorem 2.5.

Comments

(i) *Page 196: normal curvature* The normal curvature of M in the direction of a unit vector $\mathbf{u}$ tangent to M is defined to be

$$k(\mathbf{u}) = S(\mathbf{u})\cdot\mathbf{u}.$$

Note that, unlike the curvature of a curve, the normal curvature of a surface in a particular direction may be negative.

(ii) *Page 201:* $\overline{T}_p(M)$ $\overline{T}_p(M)$ was defined in Exercise IV.3.9 on page 150 to be the plane in E^3 consisting of all points of the form p + v for v in $T_p(M)$. It is the plane through p orthogonal to U(p), and gives a concrete realization of $T_p(M)$.

(iii) *Pages 197 and 201: normal sections and quadratic approximations* O'Neill makes several assumptions in these sections which he does not justify. For the sake of completeness we here point out which results he is assuming and indicate how they may be proved.

(a) Firstly, O'Neill assumes that the position of M in E^3 is of no importance, in other words that a Euclidean isometry F (as defined in Chapter III) maps principal vectors of M to principal vectors of F(M) and makes no difference to the normal and principal curvatures. Since all of these concepts are defined in terms of the shape operator, it is enough to show that F preserves the shape operator; that is,

$$S(F_*(v)) = F_*(S(v))$$

for all vectors v tangent to M. This is in fact true: the proof is given in Section VI.8, and there is no need to read it until you reach that section. Because of this result, we can always simplify our investigation of a neighbourhood of a point p on a surface M by

(i) applying a translation to move p to the origin;

(ii) applying a rotation so that the tangent plane $\overline{T}_p(M)$ becomes the xy plane and any *one* unit vector in $\overline{T}_p(M)$ becomes (1, 0, 0).

(b) Secondly, O'Neill assumes that, if M is transformed so that p becomes the origin and $\overline{T}_p(M)$ becomes the xy plane, then there is a neighbourhood of 0 in which M can be expressed as M:z = f(x, y) for some (differentiable) function f. We can in fact prove this by showing that the projection $\pi : M \longrightarrow E^2$ given by

$$\pi(p_1, p_2, p_3) = (p_1, p_2)$$

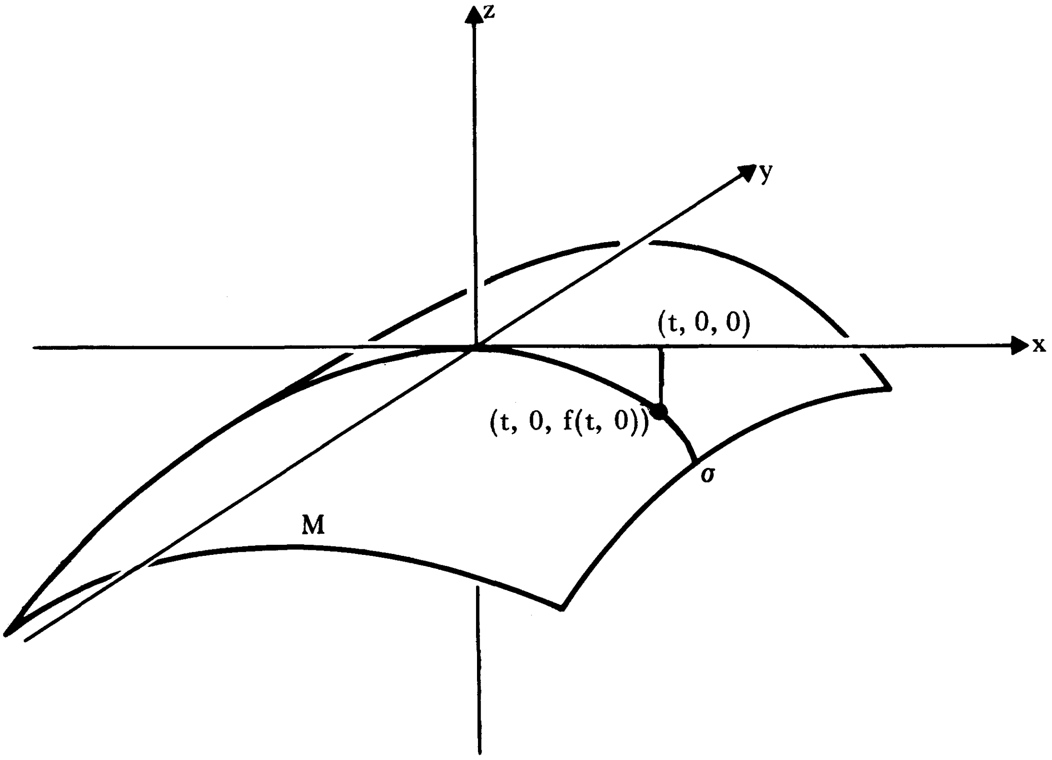
is a diffeomorphism in a neighbourhood of 0 and so has an inverse of the form

$$\pi^{-1}(p_1, p_2) = (p_1, p_2, f(p_1, p_2)).$$

The details of this proof are a little tedious, and you do not need to know them, so we omit them.

(c) Thirdly, O'Neill assumes that the normal section σ, defined on page 197, is genuinely a curve in M, and moreover has a unit-speed reparametrization. Specifically, p is a point of M, u a unit tangent vector in $T_p(M)$, and P is the plane determined by u and U(p); that is, P is the plane through p consisting of all points p + v where v is any linear combination of u and U(p). The normal section σ is defined to be $M \cap P$, and we wish to show that σ is the route of a regular curve in M.

Now, by result (a) we may assume that p is 0, $\overline{T}_p(M)$ is the xy plane (and so $U(p) = \pm (0, 0, 1)$) and u is (1, 0, 0). Then P becomes the plane y = 0. By result (b) there is a neighbourhood of 0 on which M is given by M:z = f(x, y). Hence σ is (t, 0, f(t, 0)) on an interval I of the x-axis.



The function

$$t \longmapsto (t, 0, f(t, 0))$$

is differentiable, so σ is a curve in M. Moreover, its velocity is

$$t \longmapsto (1, 0, f_x(t, 0)),$$

which is never zero: thus σ is regular and so may be given a unit-speed reparametrization.

(iv) *Page 200: Proof of Theorem 2.5* With the definitions given in the Introduction in the Text, this theorem becomes trivial, and so it suffices to prove that the Text definitions of principal curvature, principal vector, principal direction and umbilic are equivalent to those given by O'Neill. This has been done in Theorem V.2.A, so you may omit the proof given here, which makes no appeal to linear algebra.

The formula obtained in Corollary 2.6 is found during the proof of both Theorem 2.5 and Theorem V.2.A.

(v) *Page 202: quadratic approximation of f near (0, 0)* For general f, this approximation is

$$f(x, y) \sim f(0, 0) + f_x(0, 0)x + f_y(0, 0)y +$$

$$\tfrac{1}{2}(f_{xx}(0, 0)x^2 + 2f_{xy}(0, 0)xy + f_{yy}(0, 0)y^2).$$

This is the two-dimensional version of Taylor's approximation, and may be found in *Unit M201 14, Bilinear and Quadratic Forms,* page 25. We shall always use the quadratic approximation $\frac{1}{2}(k_1x^2 + k_2y^2)$, as obtained by shifting the origin and rotating the axes.

Supplementary Comments

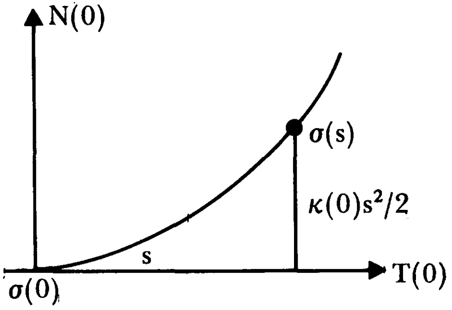
(i) *Page 197: lines -10 to -8* These lines are illustrated by Fig. 5.11 on page 198. The curve σ lies in the plane P and so the tangent vectors $\sigma'(0)$ and

$\sigma''(0)$ both lie in P. However, $\sigma'(0) = T(0)$ and $\sigma''(0) = \kappa(0)N(0)$ and so the tangent vectors T(0) and N(0) both lie in P. Moreover, σ is a curve in M and so $\sigma'(0)$ lies in $T_p(M)$. The only unit tangent vectors at p which lie in both P and $T_p(M)$ are $\pm\mathbf{u}$ and so $\sigma'(0) = \pm\mathbf{u}$. In fact we may choose σ so that $\sigma'(0) = \mathbf{u}$. Now the tangent vector N(0) is a unit vector which lies in the plane P and which is orthogonal to the tangent vector $T(0) = \sigma'(0) = \mathbf{u}$: the only such vectors are $\pm U(p)$ and so $N(0) = \pm U(p)$.

(ii) *Page 198: shape of σ* The Frenet approximation of a unit speed curve (page 61) tells us that, near 0,

$$\begin{aligned}\sigma(s) &\sim \sigma(0) + s\,T(0) + \kappa(0)\frac{s^2}{2}N(0) + \kappa(0)\tau(0)\frac{s^3}{6}B(0)\\ &= \sigma(0) + s\,T(0) + \kappa(0)\frac{s^2}{2}N(0)\end{aligned}$$

because $\tau(0) = 0$ as σ is a plane curve (lying in the plane P).



$\kappa(0)$ is non-negative by definition, and so σ bends towards N(0) on either side of $\sigma(0)$.

(iii) *Page 198: saddle surface* Since this is given by $z = xy$, along the line $y = x$ we have $z = x^2$ and so the normal section is a parabola bending upwards; along the line $y = -x$ we have $z = -x^2$ and so the normal section is a parabola bending downwards.

(iv) *Page 202: line 1* We are assuming (1) that $p \in M$ is the origin, and so $0 = f(0, 0)$; the Monge patch

$$(u, v) \longmapsto (u, v, f(u, v))$$

has partial velocities $(1, 0, f_u(0, 0))$ and $(0, 1, f_v(0, 0))$ at the origin, and by assumption (2) these are orthogonal to $(0, 0, 1)$ and so

$$f_u(0, 0) = f_v(0, 0) = 0.$$

Summary

Notation

$k(\mathbf{u})$	Page 196, Definition 2.2
k_1, k_2	Page 199, Definition 2.3 and Text, page 19
$\mathbf{e}_1, \mathbf{e}_2$	Text, page 19
$\bar{T}_p(M)$	Page 201, line −4
$\hat{M}$	Page 202, line 15

Definitions

(i)	Normal curvature ($k(u)$) in the u direction	Page 196, Definition 2.2
(ii)	Normal section σ	Page 197, line -11
(iii)	Principal curvatures k_1, k_2	Text, page 19
(iv)	Principal directions	Text, page 19
(v)	Principal vectors e_1, e_2	Text, page 19
(vi)	Umbilic point	Text, page 19
(vii)	Tangent plane $\overline{T}_p(M)$	Page 201, line -4
(viii)	Quadratic approximation ($\hat{M}$) of M near p	Page 202, line 15

Results

(i)	For a curve α in M, $\alpha''\cdot U = S(\alpha')\cdot\alpha'$.	Page 196, Lemma 2.1
(ii)	For a unit-speed curve α in M with $\alpha'(0) = u_p$: $k(u) = \kappa(0)N(0)\cdot U(p)$.	Page 197, line 8
(iii)	For a unit-speed normal section σ, with $\sigma'(0) = u_p$: $k(u) = \pm\kappa_\sigma(0)$.	Page 197, line -7
(iv)	For a unit-speed normal section σ with $\sigma'(0) = u_p$: $k(u) > 0$ implies σ is bending towards U(p) at p, $k(u) < 0$ implies σ is bending away from U(p) at p.	Page 198, first two paragraphs
(v)	The principal curvatures at p are the minimum and maximum values of the normal curvature at p.	Text, page 20, Theorem V.2.A
(vi)	The normal curvature at p is constant if and only if p is umbilic.	Text, page 20 Theorem V.2.A
(vii)	If p is umbilic, all directions at p are principal directions.	Page 200, Theorem 2.5.(1)
(viii)	If p is not umbilic, there are exactly two principal directions at p and these are orthogonal.	Page 200, Theorem 2.5.(2)
(ix)	If e_1, e_2 are principal vectors at p with corresponding principal curvatures k_1 and k_2, and $u = \cos\vartheta\, e_1 + \sin\vartheta\, e_2$, then $k(u) = k_1\cos^2\vartheta + k_2\sin^2\vartheta$.	Page 201, Corollary 2.6
(x)	If k_1, k_2 are the principal curvatures at p, the shape of M near p is approximately the same as the shape of $\hat{M} : z = \frac{1}{2}(k_1x^2 + k_2y^2)$ near **0**.	Page 202, line 15

Techniques

(i)	Finding the principal curvatures and vectors in simple cases.	Text, page 20
(ii)	Finding the quadratic approximation to a surface near a point.	Page 202, line 15

Exercises

Technique (i)

1. Page 202, Exercise 1.

Technique (ii)

2. Page 203, Exercise 4. Describe the appearance of each surface near the origin, using a sketch if possible.

Solutions

1. Page 202, Exercise 1.

(a) At any point p of the cylinder let e_1 be a unit tangent vector along a ruling of the cylinder, e_2 a unit tangent vector tangent to the cross-sectional circle through p, as in Example V.1.3.(3) on page 192. It was shown on page 192 that

$$S(e_1) = 0 = 0e_1$$

$$S(e_2) = -\frac{1}{r}e_2:$$

thus e_1 and e_2 are eigenvectors of S with eigenvalues 0 and $-1/r$. In other words, e_1 and e_2 are principal vectors at p and 0 and $-1/r$ are the principal curvatures at p.

(b) As on page 192, we let $u_1 = U_1(0)$, $u_2 = U_2(0)$. It was shown in the solution to Exercise V.1.3 that

$$S(au_1 + bu_2) = bu_1 + au_2.$$

Thus $au_1 + bu_2$ is a principal vector at p if and only if $bu_1 + au_2$ is a multiple of $au_1 + bu_2$, which happens if and only if either $a = b$ (in which case the corresponding principal curvature is 1) or $a = -b$ (in which case the corresponding principal curvature is -1).

2. Page 203, Exercise 4.

In each case M is the surface $M{:}z = f(x, y)$ for some function f. We can show that $f(0, 0) = f_x(0, 0) = f_y(0, 0) = f_{xy}(0, 0) = 0$, so that the working on page 202 tells us that $k_1 = f_{xx}(0, 0)$, $k_2 = f_{yy}(0, 0)$ and the quadratic approximation is

$$\hat{M} : z = \tfrac{1}{2}(f_{xx}(0, 0)x^2 + f_{yy}(0, 0)y^2).$$

(a) In this case $f(x, y) = \exp(x^2 + y^2) - 1$.

$$f_x(x, y) = 2x \exp(x^2 + y^2) \qquad f_{xx}(x, y) = (2 + 4x^2)\exp(x^2 + y^2)$$
$$f_y(x, y) = 2y \exp(x^2 + y^2) \qquad f_{xy}(x, y) = 4xy \exp(x^2 + y^2)$$
$$f_{yy}(x, y) = (2 + 4y^2)\exp(x^2 + y^2)$$

The quadratic approximation is the surface

$$\widehat{M} : z = \tfrac{1}{2}(2x^2 + 2y^2) = x^2 + y^2,$$

which is a paraboloid bending upwards with its "south pole" at the origin, so M is bowl-shaped near **0**. For a sketch, invert Fig. 5.18 on page 204.

(b) Now $f(x, y) = \log \cos x - \log \cos y$.

$$f_x(x, y) = -\tan x \qquad f_{xx}(x, y) = -\sec^2 x$$
$$f_y(x, y) = \tan y \qquad f_{xy}(x, y) = 0$$
$$f_{yy}(x, y) = \sec^2 y$$

The quadratic approximation is the surface

$$\widehat{M} : z = \tfrac{1}{2}(-x^2 + y^2) = \tfrac{1}{2}(y^2 - x^2),$$

which is a saddle surface (it cuts the xz plane in the parabola $z = -\frac{1}{2}x^2$, which bends downwards, and the yz plane in the parabola $z = \frac{1}{2}y^2$, which bends upwards) so M is saddle-shaped near **0**. See Fig. 5.19 on page 204.

(c) Now $f(x, y) = (x + 3y)^3$

$$f_x(x, y) = 3(x + 3y)^2 \qquad f_{xx}(x, y) = 6(x + 3y)$$
$$f_y(x, y) = 9(x + 3y)^2 \qquad f_{xy}(x, y) = 18(x + 3y)$$
$$f_{yy}(x, y) = 54(x + 3y)$$

The quadratic approximation is the surface

$$\widehat{M} : z = \tfrac{1}{2}(0x^2 + 0y^2),$$

that is, the xy plane. In fact this tells us nothing about the shape of M near **0** without further information.

V.3 GAUSSIAN CURVATURE

Introduction

This section depends only on Section V.2 and the following two vector identities.

Lemma V.3.A If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are vectors in E^3 then

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}.$$

This occurs in *Unit MST 282 1, Some Basic Tools*, Section 2.3. We use it to prove the next lemma.

Lemma V.3.B (The Lagrange Identity). If $\mathbf{v}, \mathbf{w}, \mathbf{a}, \mathbf{b}$ are vectors in E^3 then

$$(\mathbf{v} \times \mathbf{w})\cdot(\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} \mathbf{v}\cdot\mathbf{a} & \mathbf{v}\cdot\mathbf{b} \\ \mathbf{w}\cdot\mathbf{a} & \mathbf{w}\cdot\mathbf{b} \end{vmatrix}$$

Proof Writing $\mathbf{a} \times \mathbf{b} = \mathbf{c}$ we have

$$\begin{aligned}(\mathbf{v} \times \mathbf{w})\cdot(\mathbf{a} \times \mathbf{b}) &= (\mathbf{v} \times \mathbf{w})\cdot\mathbf{c} \\ &= \mathbf{v}\cdot(\mathbf{w} \times \mathbf{c}), && \text{by Exercise II.1.4(d),} \\ &= \mathbf{v}\cdot(\mathbf{w} \times (\mathbf{a} \times \mathbf{b})) \\ &= \mathbf{v}\cdot((\mathbf{w}\cdot\mathbf{b})\mathbf{a} - (\mathbf{w}\cdot\mathbf{a})\mathbf{b}), && \text{by Lemma V.3.A,} \\ &= (\mathbf{v}\cdot\mathbf{a})(\mathbf{w}\cdot\mathbf{b}) - (\mathbf{v}\cdot\mathbf{b})(\mathbf{w}\cdot\mathbf{a}) \\ &= \begin{vmatrix} \mathbf{v}\cdot\mathbf{a} & \mathbf{v}\cdot\mathbf{b} \\ \mathbf{w}\cdot\mathbf{a} & \mathbf{w}\cdot\mathbf{b} \end{vmatrix}. \end{aligned}$$

Having found the geometric significance of the eigenvectors and eigenvalues of S in the previous section, we now investigate that of the determinant and trace of S. Recall that the determinant of a transformation with matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is $ad - bc$, and that the trace of this transformation is $a + d$.

Both of these quantities are independent of the choice of basis.

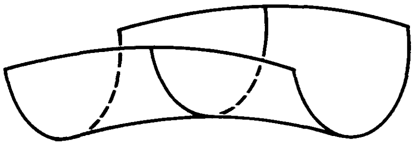
READ: Section V.3 (pages 203-207).

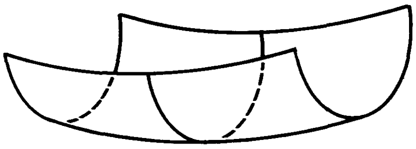
Comments

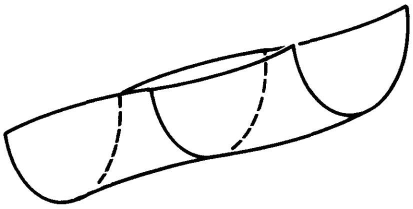
(i) *Page 203: Gaussian and mean curvature* If U is replaced by −U, the Gaussian curvature K is unchanged and so K can be defined unambiguously over the whole of a surface M, even if M is not orientable. The mean curvature H has the usual ambiguity of sign and so cannot be defined over the whole of a nonorientable surface.

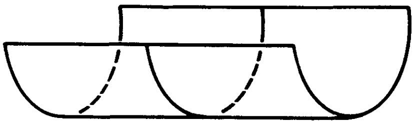
(ii) *Pages 204-205: Remark 3.3* The monkey saddle exemplifies a fact which is implicit in this remark, that if a surface is not of types (1), (2) or (3(a)) (that is, bowl-shaped, saddle-shaped or trough-shaped) near p then p *must* be a planar point; that is, $k_1(p) = k_2(p) = 0$.

The illustration, Fig. 5.20, for the case (3(a)) ($k_1(p) \neq 0$, $k_2(p) = 0$) is a little misleading, as we have no information about how M is curving in the e_2 direction. Each of the diagrams below shows a possibility.









The difficulty is akin to that of distinguishing between the types of stationary point t of a function $g: R \longrightarrow R$ when $g'(t) = g''(t) = 0$.

Supplementary Comments

(i) *Page 206: proof of second half of Lemma 3.4*

$$\begin{aligned} S(v) \times w + v \times S(w) &= (av + bw) \times w + v \times (cv + dw) \\ &= av \times w + dv \times w \\ &= 2H(p)\, v \times w. \end{aligned}$$

(ii) *Page 206: proof of formulas in centre of page* Dotting each side of

$$S(V) \times S(W) = K\, V \times W$$

with $V \times W$ gives

$$(S(V) \times S(W))\cdot(V \times W) = K(V \times W)\cdot(V \times W),$$

which, by the Lagrange identity, is

$$\begin{vmatrix} S(V)\cdot V & S(V)\cdot W \\ S(W)\cdot V & S(W)\cdot W \end{vmatrix} = K \begin{vmatrix} V\cdot V & V\cdot W \\ W\cdot V & W\cdot W \end{vmatrix}.$$

The other formula is obtained similarly.

O'Neill is writing SV for S(V). He frequently drops the parentheses when the meaning is unambiguous—for example he writes Cv for C(v) in Chapter III and fx for f(x) in Chapter IV.

(iii) *Page 207: singular* Since S has matrix

$$\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

with respect to a suitable basis, S is singular if and only if at least one of k_1, k_2 is zero, that is if and only if $\det S = k_1 k_2 = K = 0$.

Summary

Notation

K	Page 203, Definition 3.1
H	Page 203, line 7 of the section

Definitions

(i)	Gaussian curvature K	Page 203, Definition 3.1
(ii)	Mean curvature H	Page 203, line 7 of the section
(iii)	Planar point	Page 205, line 4
(iv)	Flat surface	Page 207, Definition 3.6
(v)	Minimal surface	Page 207, Definition 3.6

Results

(i)	$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$.	Text, page 27
(ii)	$(v \times w) \cdot (a \times b) = \begin{vmatrix} v \cdot a & v \cdot b \\ w \cdot a & w \cdot b \end{vmatrix}$	Text, page 27
(iii)	$K = k_1 k_2$; $H = (k_1 + k_2)/2$.	Page 203, Lemma 3.2
(iv)	$K(p) > 0 \Rightarrow$ M is bowl-shaped near p; $K(p) < 0 \Rightarrow$ M is saddle-shaped near p; $K(p) = 0 \Rightarrow$ p is a planar point or M is trough-shaped near p.	Page 204, Remark 3.3
	M has a complicated shape near p $\Rightarrow$ p is a planar point.	Text, page 28
(v)	If v and w span $T_p(M)$, then $S(v) \times S(w) = K(p)\, v \times w$ $S(v) \times w + v \times S(w) = 2H(p)\, v \times w$.	Page 205, Lemma 3.4

(vi) If V, W are tangent vector fields which are linearly independent at each point, then

$$K = \frac{\begin{vmatrix} SV\cdot V & SV\cdot W \\ SW\cdot V & SW\cdot W \end{vmatrix}}{\begin{vmatrix} V\cdot V & V\cdot W \\ W\cdot V & W\cdot W \end{vmatrix}}$$

$$H = \frac{\begin{vmatrix} SV\cdot V & SV\cdot W \\ W\cdot V & W\cdot W \end{vmatrix} + \begin{vmatrix} V\cdot V & V\cdot W \\ SV\cdot V & SV\cdot W \end{vmatrix}}{2\begin{vmatrix} V\cdot V & V\cdot W \\ W\cdot V & W\cdot W \end{vmatrix}}$$

Page 206, line 15

(vii) $k_1, k_2 = H \pm (H^2 - K)^{\frac{1}{2}}$. Page 206, Corollary 3.5

Techniques

(i) Finding K and H in simple cases. Page 203, Definition 3.1 and line 7 of the section

(ii) Interpreting K and H geometrically. Page 204, Remark 3.3

Exercises

Technique (i)

1. Find K(0) and H(0) for each of the surfaces in Exercise 4 on page 203.

Technique (ii)

2. Page 207, Exercise 1.

3. Describe the shape of M near p if exactly one of K(p), H(p) is zero.

Theory Exercise

4. Page 207, Exercise 3. (HINT: Use Corollary V.2.6.)

Solutions

1. Writing each of these surfaces as M:z = f(x, y) in turn, we found in the previous section that

$$f(0, 0) = f_x(0, 0) = f_y(0, 0) = f_{xy}(0, 0) = 0.$$

Thus the result of Exercise V.1.2 on page 194 shows that $\mathbf{u}_1$ and $\mathbf{u}_2$ are principal vectors with corresponding principal curvatures $f_{xx}(0, 0)$ and $f_{yy}(0, 0)$. Thus $K(0) = f_{xx}(0, 0)\, f_{yy}(0, 0)$ and $H(0) = \frac{1}{2}(f_{xx}(0, 0) + f_{yy}(0, 0))$. We have already calculated $f_{xx}(0, 0)$ and $f_{yy}(0, 0)$ in these three cases.

(a) $f(x, y) = \exp(x^2 + y^2) - 1$

$f_{xx}(0, 0) = 2$

$f_{yy}(0, 0) = 2$

$K(\mathbf{0}) = 4; H(\mathbf{0}) = 2.$

(b) $f(x, y) = \log \cos x - \log \cos y$

$f_{xx}(0, 0) = -1$

$f_{yy}(0, 0) = 1$

$K(\mathbf{0}) = -1; H(\mathbf{0}) = 0.$

(c) $f(x, y) = (x + 3y)^3$

$f_{xx}(0, 0) = 0$

$f_{yy}(0, 0) = 0$

$K(\mathbf{0}) = 0; H(\mathbf{0}) = 0.$

2. Page 207, Exercise 1.

If $K(\mathbf{p}) < 0$, $k_1(\mathbf{p})$ and $k_2(\mathbf{p})$ must have opposite signs and so cannot be equal: hence $\mathbf{p}$ is not umbilic. If $K(\mathbf{p}) \leqslant 0$ and $\mathbf{p}$ is umbilic, then $k_1(\mathbf{p}) = k_2(\mathbf{p})$ and $k_1(\mathbf{p})k_2(\mathbf{p}) \leqslant 0$, which forces

$$k_1(\mathbf{p}) = k_2(\mathbf{p}) = 0;$$

that is, $\mathbf{p}$ is planar.

3. If $K(\mathbf{p}) = 0$ then at least one of $k_1(\mathbf{p})$, $k_2(\mathbf{p})$ is zero. Were they both zero, $H(\mathbf{p})$ would also be zero. This is not so, so exactly one of the principal curvatures is zero and M is trough-shaped near $\mathbf{p}$.

If $H(\mathbf{p}) = 0$ then $k_1(\mathbf{p})$ and $k_2(\mathbf{p})$ have opposite signs so $K(\mathbf{p}) \leqslant 0$. $K(\mathbf{p})$ is nonzero, so $K(\mathbf{p}) < 0$, so M is saddle-shaped near $\mathbf{p}$.

4. Page 207, Exercise 3.

(a) If $\mathbf{u}$ is a unit vector in $T_{\mathbf{p}}(M)$ we can write

$$\mathbf{u} = \cos\vartheta\, \mathbf{e}_1 + \sin\vartheta\, \mathbf{e}_2$$

for some ϑ, where $\mathbf{e}_1, \mathbf{e}_2$ are orthogonal principal vectors at $\mathbf{p}$.

If $\mathbf{v}$ is a unit vector in $T_{\mathbf{p}}(M)$ orthogonal to $\mathbf{u}$ then

$$\mathbf{v} = \cos\varphi\, \mathbf{e}_1 + \sin\varphi\, \mathbf{e}_2$$

where $\varphi = \vartheta + \frac{\pi}{2}$ or $\varphi = \vartheta - \frac{\pi}{2}$.

By Corollary V.2.6,

$$k(\mathbf{u}) = k_1 \cos^2\vartheta + k_2 \sin^2\vartheta$$

$$k(\mathbf{v}) = k_1 \cos^2\varphi + k_2 \sin^2\varphi.$$

Now

$$\cos\varphi = \cos(\vartheta + \tfrac{\pi}{2}) \quad \text{or} \quad \cos(\vartheta - \tfrac{\pi}{2})$$

$$= -\sin\vartheta \quad \text{or} \quad \sin\vartheta$$

so

$$\cos^2\varphi = \sin^2\vartheta;$$

while

$$\sin\varphi = \sin(\vartheta + \tfrac{\pi}{2}) \quad \text{or} \quad \sin(\vartheta - \tfrac{\pi}{2})$$

$$= \cos\vartheta \quad \text{or} \quad -\cos\vartheta$$

so

$$\sin^2\varphi = \cos^2\vartheta.$$

Thus

$$k(v) = k_1\sin^2\vartheta + k_2\cos^2\vartheta$$

and so

$$\begin{aligned}\tfrac{1}{2}(k(u) + k(v)) &= \tfrac{1}{2}(k_1(\cos^2\vartheta + \sin^2\vartheta) + k_2(\sin^2\vartheta + \cos^2\vartheta)) \\ &= \tfrac{1}{2}(k_1 + k_2) \\ &= H(p).\end{aligned}$$

(b) If $k(\vartheta)$ denotes $k(\cos\vartheta\, e_1 + \sin\vartheta\, e_2)$,

$$\begin{aligned}\int_0^{2\pi} k(\vartheta)d\vartheta &= \int_0^{2\pi}(k_1\cos^2\vartheta + k_2\sin^2\vartheta)d\vartheta \\ &= \pi k_1 + \pi k_2 \\ &= \pi(k_1 + k_2) \\ &= 2\pi H(p).\end{aligned}$$

V.4 COMPUTATIONAL TECHNIQUES

Introduction

This section follows Section V.3 and completes the block introducing the theory of shape. Applications are dealt with in Sections V.5 and V.6.

READ: Section V.4 (pages 210-219).

Comments

(i) *Partial differentiation* In Section IV.2 we saw how partial differentiation came into the definition of the partial velocities x_u and x_v. Now we have encountered their partial derivatives x_{uu}, x_{vv} and $x_{uv} = x_{vu}$ and need to be quite clear how they arise. For instance, the u-parameter curve $v = v_0$ is given by

$$t \longmapsto x(t, v_0).$$

Its velocity vector field is given by

$$t \longmapsto x_u(t, v_0) = \left(\frac{\partial x_1}{\partial u}(t, v_0), \frac{\partial x_2}{\partial u}(t, v_0), \frac{\partial x_3}{\partial u}(t, v_0)\right)_{x(t,\, v_0)}.$$

The derivative of this vector field, the acceleration of the u- parameter curve, is

$$t \longmapsto \left(\frac{\partial^2 x_1}{\partial u^2}(t, v_0), \frac{\partial^2 x_2}{\partial u^2}(t, v_0), \frac{\partial^2 x_3}{\partial u^2}(t, v_0)\right)_{x(t,\, v_0)} = x_{uu}(t, v_0).$$

Hence the function x_{uu} gives the acceleration of each u-parameter curve at each point. That is, $x_{uu}(u_0, v_0)$ is the acceleration of the u-parameter curve $v = v_0$ when $u = u_0$ — that is, at the point $x(u_0, v_0)$.

Similar interpretations can be given for x_{vv} and $x_{uv} = x_{vu}$.

We have also used the partial derivative U_v. Now U is a function defined on the domain of the patch x whose value at (u_0, v_0) is a tangent vector normal to the surface at $x(u_0, v_0)$. Hence U is a function

$$U : (u, v) \longmapsto (f(u, v), g(u, v), h(u, v))_{x(u,\, v)},$$

where f, g and h are real valued functions on the domain of x. We form the partial derivative U_v by taking the partial derivative of each coordinate; that is,

$$U_v : (u, v) \longmapsto \left(\frac{\partial f}{\partial v}(u, v), \frac{\partial g}{\partial v}(u, v), \frac{\partial h}{\partial v}(u, v)\right)_{x(u,\, v)}.$$

It is possible to define this partial derivative in a coordinate independent way. The composite function $U \circ x^{-1}$ gives us a tangent vector field on the image of the patch and hence we can define the directional derivative $\nabla_{x_v}(U \circ x^{-1})$. But

$$\nabla_{x_v}(U \circ x^{-1}) = (x_v[f \circ x^{-1}], x_v[g \circ x^{-1}], x_v[h \circ x^{-1}])_x$$

$$= \left(\frac{\partial}{\partial v}(f \circ x^{-1} \circ x), \frac{\partial}{\partial v}(g \circ x^{-1} \circ x), \frac{\partial}{\partial v}(h \circ x^{-1} \circ x)\right)_x,$$

by Exercise IV.3.4(b),

$$= \left(\frac{\partial f}{\partial v}, \frac{\partial g}{\partial v}, \frac{\partial h}{\partial v}\right)_x = U_v.$$

Hence $U_v = \nabla_{x_v}(U \circ x^{-1})$, which is defined in a coordinate independent way.

When we partially differentiate the dot product of vector valued functions U and V we obtain the usual Leibnizian result

$$\frac{\partial}{\partial v}(U \cdot V) = U_v \cdot V + U \cdot V_v.$$

Now, in Section V.1 we came across the result $S(\alpha') = -U'$, where α was a curve in the surface and U was the restriction of a normal vector field to the curve. On page 212 we want to look at what happens along the v-parameter curve $u = u_0$ given by

$$\alpha : t \longmapsto x(u_0, t).$$

We know that the velocity vector field is given by

$$\alpha' : t \longmapsto x_v(u_0, t)$$

and the restriction of the normal vector field given on page 211 is

$$U : t \longmapsto U(u_0, t).$$

The derivative of this last vector field is hence

$$t \longmapsto U_v(u_0, t)$$

and we have that

$$S(x_v(u_0, t)) = -U_v(u_0, t),$$

for all u_0 and t, and hence

$$S(x_v) = -U_v$$

as required.

Alternatively we could have obtained this result from first principles. A normal vector field on the image of the patch x is given by the composite function $U \circ x^{-1}$. Hence by the defintion of S

$$S(x_v(u_0, v_0)) = -\nabla_{x_v(u_0, v_0)}(U \circ x^{-1}).$$

But $x_v(u_0, v_0)$ is the initial velocity of the curve

$$t \longmapsto x(u_0, v_0 + t)$$

and hence

$$\begin{aligned} S(x_v(u_0, v_0)) &= -\frac{d}{dt}(U \circ x^{-1} \circ x(u_0, v_0 + t))\Big|_{t=0} \\ &= -\frac{d}{dt}(U(u_0, v_0 + t))\Big|_{t=0} \\ &= -U_v(u_0, v_0), \end{aligned}$$

which is the required result.

(ii) *S is symmetric* To prove that S is symmetric we need to prove that $S(v) \cdot w = S(w) \cdot v$ for each pair v, w of tangent vectors to M at $p = x(u_0, v_0)$. But $v = ax_u + bx_v$ and $w = cx_u + dx_v$ for some real numbers a, b, c, d, where as usual x_u and x_v are evaluated at (u_0, v_0). It follows that

$$\begin{aligned} S(v) \cdot w &= S(ax_u + bx_v) \cdot (cx_u + dx_v) \\ &= (aS(x_u) + bS(x_v)) \cdot (cx_u + dx_v) \\ &= ac\, S(x_u) \cdot x_u + (bc + ad)\, S(x_u) \cdot x_v + bd\, S(x_v) \cdot x_v, \end{aligned}$$

since $S(x_u) \cdot x_v = S(x_v) \cdot x_u$. Similarly

$$S(w) \cdot v = ac\, S(x_u) \cdot x_u + (bc + ad) S(x_u) \cdot x_v + bd\, S(x_v) \cdot x_v = S(v) \cdot w.$$

Since there are enough patches to cover all of the surface we have proved that S is symmetric.

(iii) *Lemma 4.4 and Example 4.5* We need to use the triple scalar product. Given (tangent) vectors **a**, **b** and **c** the triple scalar product is $\mathbf{a}\cdot\mathbf{b} \times \mathbf{c}$. From Section II.1 we know that the triple scalar product is zero if any two of the vectors **a**, **b** and **c** are scalar multiples of each other. This enables us to prove Lemma 4.4 directly.

By Lemma V.3.4,

$$S(V) \times S(W) = KV \times W = KZ.$$

Hence forming the dot product of both sides with Z we obtain

$$Z\cdot\left(\frac{-\nabla_V Z}{\|Z\|} - V\left[\frac{1}{\|Z\|}\right]Z\right) \times \left(\frac{-\nabla_W Z}{\|Z\|} - W\left[\frac{1}{\|Z\|}\right]Z\right) = K\|Z\|^2.$$

Expanding the left hand side as a linear combination of scalar triple products we find only one term in which the three vectors are distinct and hence

$$\frac{Z\cdot(\nabla_V Z \times \nabla_W Z)}{\|Z\|^2} = K\|Z\|^2,$$

from which the result follows. The formula for H follows similarly.

In Section II.1 we saw further that the triple scalar product can be expressed as a determinant. We know that determinants are linear in each row, from which the equalities on page 218, line 3 follow.

Summary

Notation

E, F, G	Page 210, line 11
U	Page 211, line 3
U_V	Text, page 33
x_{uu}, x_{uv}, x_{vv}	Text, page 33
ℓ, m, n	Page 211, lines -9 to -7
W	Page 214, line 6
h	Page 218, line -6

Definitions

(i)	The speed ($\sqrt{E}, \sqrt{G}$) of the parameter curves	Page 210, line 11
(ii)	The unit normal function U	Page 211, line 3
(iii)	The acceleration (x_{uu} and x_{vv}) of the parameter curves	Page 211, lines 16 and 18, and Text, page 33
(iv)	The support function h	Page 218, line -6

Results

(i)	$K=\dfrac{\ell n-m^2}{EG-F^2}$, $H=\dfrac{G\ell+En-2Fm}{2(EG-F^2)}$.	Page 212, Corollary 4.1.
(ii)	S is symmetric.	Page 212, line -2 to Page 213, line 2
(iii)	$\ell = U\cdot x_{uu}$, $m = U\cdot x_{uv}$, $n = U\cdot x_{vv}$.	Page 213, Lemma 4.2
(iv)	If Z is a nonvanishing normal vector field and $V \times W = Z$ then $K=\dfrac{(Z\cdot\nabla_V Z \times \nabla_W Z)}{\lVert Z\rVert^4}$ $H=-Z\cdot\dfrac{(\nabla_V Z \times W + V \times \nabla_W Z)}{2\lVert Z\rVert^3}$.	Page 217, Lemma 4.4

Techniques

(i)	Calculating E, F, G, W, U, ℓ, m, n, K and H on a patch x.	Pages 213-215, Example 4.3
(ii)	Calculating K and H using linearly independent tangent vector fields.	Page 217, Lemma 4.4.

Exercises

Technique (i)

1. Page 219, Exercise 2.
2. Page 219, Exercise 3.
3. Page 219, Exercise 4.

Technique (ii)

4. Page 222, Exercise 20.

Solutions

1. Page 219, Exercise 2.

$$x(u, v) = (u, v, f(u, v))$$

$$x_u = (1, 0, f_u) \qquad E = 1 + f_u^2$$

$$x_v = (0, 1, f_v) \qquad F = f_u f_v$$

$$G = 1 + f_v^2$$

$$W = (EG - F^2)^{\frac{1}{2}} = (1 + f_u^2 + f_v^2)^{\frac{1}{2}}$$

$$U = x_u \times x_v/W = \frac{1}{W}\begin{vmatrix} U_1 & U_2 & U_3 \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = (-f_u, -f_v, 1)/W$$

$$x_{uu} = (0, 0, f_{uu}) \qquad \ell = U \cdot x_{uu} = f_{uu}/W$$

$$x_{uv} = (0, 0, f_{uv}) \qquad m = U \cdot x_{uv} = f_{uv}/W$$

$$x_{vv} = (0, 0, f_{vv}) \qquad n = U \cdot x_{vv} = f_{vv}/W$$

$$K = \frac{\ell n - m^2}{EG - F^2} = \frac{f_{uu} f_{vv} - f_{uv}^2}{W^4}$$

$$H = \frac{G\ell + En - 2Fm}{2(EG - F^2)}$$

$$= \frac{(1 + f_v^2) f_{uu} + (1 + f_u^2) f_{vv} - 2 f_u f_v f_{uv}}{2W^3}$$

2. Page 219, Exercise 3.

The image is flat if and only if $K = 0$ and minimal if and only if $H = 0$. The conditions follow from Exercise 1.

3. Page 219, Exercise 4.

For this exercise $f(u, v) = \log \cos v - \log \cos u$.

Hence $f_u = -\frac{(-\sin u)}{\cos u} = \tan u$.

Similarly $f_v = \frac{(-\sin v)}{\cos v} = -\tan v$.

Then $f_{uu} = \sec^2 u$, $f_{uv} = 0$, $f_{vv} = -\sec^2 v$.

Since $(1 + f_u^2) f_{vv} + (1 + f_v^2) f_{uu} - 2 f_u f_v f_{uv}$

$$= (1 + \tan^2 u)(-\sec^2 v) + (1 + \tan^2 v)\sec^2 u$$

$$= -\sec^2 u \sec^2 v + \sec^2 v \sec^2 u = 0,$$

the surface is minimal.

The Gaussian curvature is given by

$$K = \frac{f_{uu} f_{vv} - f_{uv}^2}{W^4}$$

$$= -\sec^2 u \sec^2 v/W^4, \qquad \text{where}$$

$$W^2 = 1 + f_u^2 + f_v^2 = 1 + \tan^2 u + \tan^2 v.$$

4. Page 222, Exercise 20.

The saddle surface is the image of the Monge patch $x(u, v) = (u, v, uv)$. In Section IV.2 we defined a ruled surface as one that could be described by a parametrization of the form

$$x(u, v) = \beta_1(u) + v\delta_1(u) \quad \text{or} \quad \beta_2(v) + u\delta_2(v).$$

It is easy to see that the saddle surface is a ruled surface as in this case

$$\begin{aligned} x(u, v) &= (u, 0, 0) + v(0, 1, u) \\ &= (0, v, 0) + u(1, 0, v) \end{aligned}$$

and so we can choose

$$\beta_1(u) = (u, 0, 0), \quad \delta_1(u) = (0, 1, u),$$

$$\beta_2(v) = (0, v, 0), \quad \delta_2(v) = (1, 0, v).$$

The rulings give us straight lines lying within the surface whose directions can be used as a tangent vector fields to M. Thus we obtain tangent vector fields, V and W, such that

$$V(x(u, v)) = (0, 1, u) \text{ at } x(u, v) = (u, v, uv)$$

$$W(x(u, v)) = (1, 0, v) \text{ at } x(u, v) = (u, v, uv).$$

Since these are always linearly independent $Z = V \times W$ is a non-vanishing normal vector field.

By definition

$$Z(x(u, v)) = \begin{vmatrix} U_1 & U_2 & U_3 \\ 0 & 1 & u \\ 1 & 0 & v \end{vmatrix} = (v, u, -1).$$

In order to use the formulas of Lemma V.4.4 we need to calculate $\nabla_V Z$ and $\nabla_W Z$. Now by definition $V(x(u, v))$ and $W(x(u, v))$ are the initial velocities of the curves

$$t \longmapsto x(u, v) + t(0, 1, u) = (u, t + v, u(t + v))$$

and

$$t \longmapsto x(u, v) + t(1, 0, v) = (t + u, v, v(t + u)),$$

i.e.

$$t \longmapsto x(u, t + v)$$

and

$$t \longmapsto x(t + u, v).$$

So

$$\begin{aligned} \nabla_{V(x(u, v))} Z &= \frac{d}{dt}(Z(x(u, t + v))\Big|_{t=0} \\ &= \frac{d}{dt}(t + v, u, -1)\Big|_{t=0} = (1, 0, 0) \end{aligned}$$

and similarly

$$\nabla_{W(x(u,\, v))}Z = (0,\, 1,\, 0).$$

Hence by Lemma V.4.4

$$K(x(u,\, v)) = \frac{(v,\, u,\, -1)\cdot(1,\, 0,\, 0) \times (0,\, 1,\, 0)}{\|(v,\, u,\, -1)\|^4}$$

$$= \frac{-1}{(1 + u^2 + v^2)^2}$$

and

$$H(x(u,\, v)) = \frac{-(v,\, u,\, -1)\cdot((1,\, 0,\, 0) \times (1,\, 0,\, v) + (0,\, 1,\, u) \times (0,\, 1,\, 0))}{2\|(v,\, u,\, -1)\|^3}$$

$$= \frac{uv}{(1 + u^2 + v^2)^{\frac{3}{2}}}$$

These are the results obtained in Example V.4.3(2) apart from the sign of H, which is different due to the fact that Z and U point in opposite directions. H does depend on such a choice while K does not.

V.5 SPECIAL CURVES IN A SURFACE

Introduction

This section follows Section V.4.

We have introduced the shape operator, used it to define normal, Gaussian and mean curvature and obtained effective computational techniques for calculating these quantities. In this section we use this apparatus to describe some special types of curves on a surface.

READ: Section V.5 (pages 223-229).

Comments

(i) *The shape operator S and the unit normal U* The fundamental relationship between the shape operator S and the unit normal U is

$$S(\alpha') = -U',$$

where α is a curve in the surface and we are considering the vector field U restricted to α. We also regularly use the relationship

$$S(\alpha')\cdot\alpha' = \alpha''\cdot U.$$

By the definition of a principal direction we have that α is a principal curve if and only if

$$S(\alpha') = k\alpha'$$

for some real valued function k. Hence principal curves are characterized by the equations $S(\alpha') = k\alpha' = -U'$ and in this case k is the corresponding principal curvature.

This leads immediately to Lemma 5.2(1), and Lemma 5.2(2) follows since

$$k = \frac{S(\alpha')\cdot\alpha'}{\alpha'\cdot\alpha'} = \frac{\alpha''\cdot U}{\alpha'\cdot\alpha'}.$$

There are certain trivial cases where principal curves occur. If for some curve α we find that $U' = 0$ (that is, U is parallel) as in parts of Lemma 5.3 and Lemma 5.6, then it follows that

$$0 = -U' = S(\alpha') = 0\alpha'.$$

Hence α is a principal curve and the corresponding principal curvature at each point is zero. Conversely if α is a principal curve with zero principal curvature then the restriction of U to α is parallel.

By definition a curve α is asymptotic if it lies in an asymptotic direction; that is, if T is the unit tangent vector field then $k(T) = S(T)\cdot T = 0$. Since $T = \alpha'/\|\alpha'\|$ and S is linear, an equivalent definition is that

$$S(\alpha')\cdot\alpha' = 0.$$

Using the identity $S(\alpha')\cdot\alpha' = \alpha''\cdot U$ we see that the curve α is asymptotic if and only if

$$0 = S(\alpha')\cdot\alpha' = \alpha''\cdot U.$$

Again there are trivial cases. If $\alpha'' = 0$ (that is, α is a straight line) then α is an asymptotic curve.

We can use this characterization to throw some light on the remarks on page 226 concerning asymptotic directions and the tangent plane. Suppose that the intersection of the tangent plane $\overline{T}_p(M)$ with M contains a curve α such that $\alpha(0) = p$. Suppose **u** is the normal to the plane: then, since α lies in the plane, we find that $\mathbf{u}\cdot\alpha' = \mathbf{u}\cdot\alpha'' = 0$. But since the plane is tangent to M at p we have that $\mathbf{u} = U(p)$. Hence

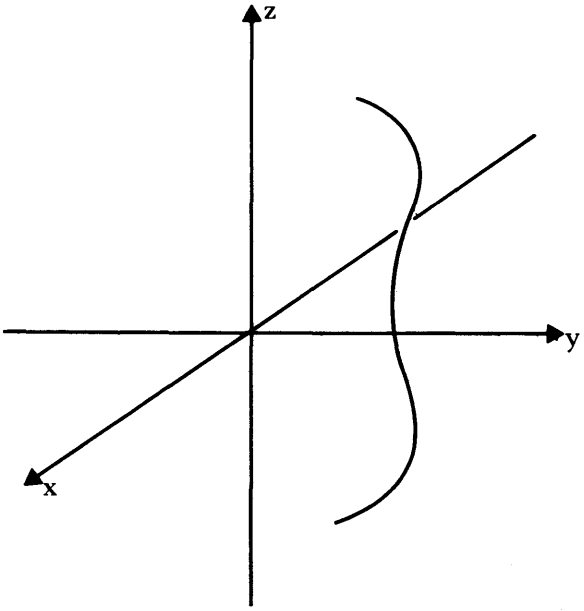
$$S(\alpha'(0))\cdot\alpha'(0) = \alpha''(0)\cdot U(p) = \alpha''(0)\cdot\mathbf{u} = 0$$

and so initially α lies in an asymptotic direction.

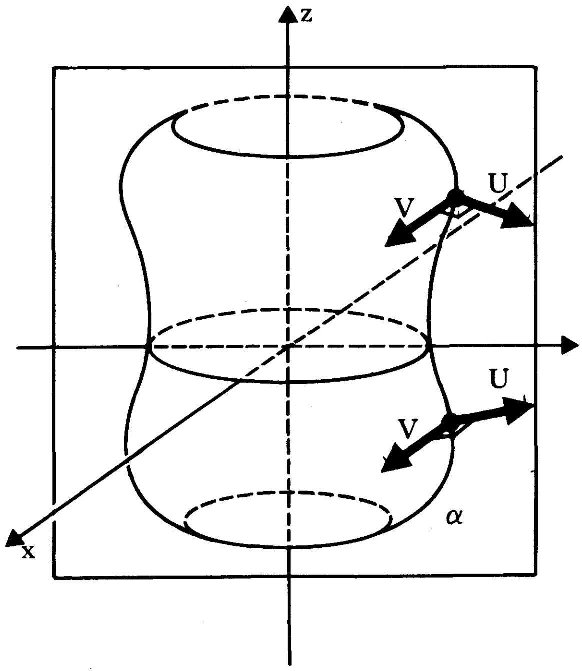
Finally, suppose a curve α is both principal and asymptotic: then $S(\alpha') = k\alpha'$ and $S(\alpha')\cdot\alpha' = 0$. Since for a (regular) curve $\alpha'\cdot\alpha'$ is never zero it follows that k, one of the principal curvatures, is zero.

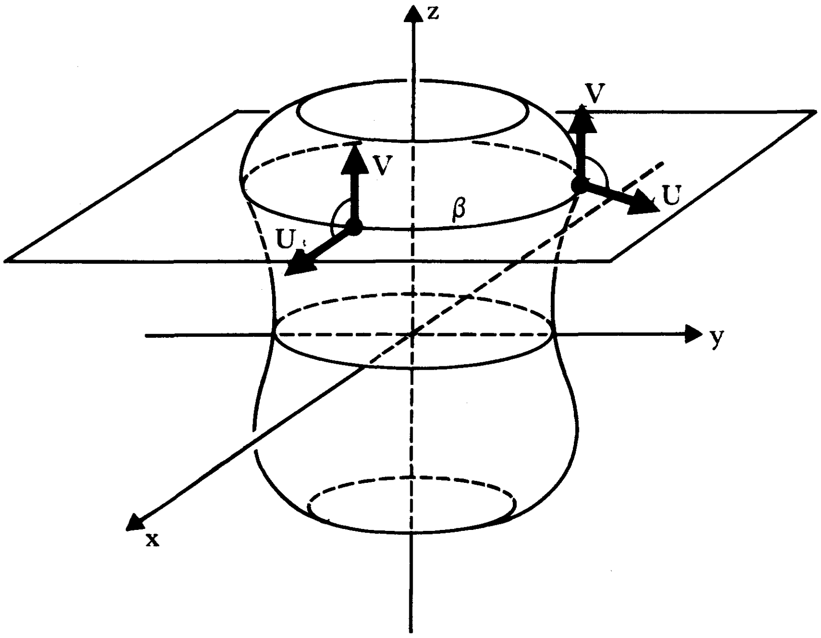
(ii) *Surfaces of Revolution and Lemma 5.3* By Lemma 5.3 we know that if a curve in a surface is the intersection of a plane with the surface and the angle between the normal to the plane and the normal to the surface is constant along the curve then the curve is principal. We can use this result to show that the meridians and parallels on a surface of revolution are principal as in the following example.

Suppose C is a suitable curve in the yz plane.



Revolving this about the z axis we obtain a surface M. The meridians are the intersection of M with vertical planes through the z axis and the parallels are the intersection of M with horizontal planes.





Along each of the curves α and β there is a constant angle between the normal vector fields U and V.

(iii) *Geodesics* In the definition of principal and asymptotic curves we restricted attention to regular curves, while in the definition of geodesics we made no such restriction. The technical reasons for this are beyond the scope of this course. However, since geodesics are constant speed curves any geodesic that is not regular is a zero-speed curve; in other words, it is a constant curve. Conversely for any constant curve α we have $\alpha'' = 0$ and hence α is a geodesic. Consequently we need to restrict attention to regular geodesics. In the examples on pages 228-229 we restrict attention to regular geodesics.

Supplementary Comment

(i) *Page 225: first four lines* If U and V are linearly independent at each point of α then there exists a never zero vector field N on α which is such that N(p) spans the tangent vector space orthogonal to both U(p) and V(p), for each point p on α. Since α' and U' are both orthogonal to U and V it follows that they are both collinear with N and hence with each other.

Summary

Definitions

(i)	Principal curve	Page 223, Definition 5.1
(ii)	Asymptotic direction	Page 225, line 16
(iii)	Asymptotic curve	Page 226, Definition 5.5
(iv)	Geodesic	Page 228, Definition 5.7
(v)	Great circle	Page 229, line 4

Examples

(i)	The geodesics on a plane are the straight lines.	Page 228, Example 5.8(1)
(ii)	The geodesics on a sphere are the great circles.	Page 228, Example 5.8(2)

Results

(i)	A curve α is principal if and only if $S(\alpha') = k\alpha' = -U'$.	Text, page 40 and Page 223, Lemma 5.2(1)
(ii)	If α is principal then the principal curvature is $\alpha''\cdot U/\alpha'\cdot\alpha'$.	Page 223, Lemma 5.2(2)
(iii)	If α is a curve cut from M by a plane P making a constant angle with the surface along α then α is a principal curve.	Page 224, Lemma 5.3
(iv)	The meridians and parallels of a surface of revolution are principal curves.	Page 225, line 9
(v)	(1) If $K(p) > 0$ there are no asymptotic directions at p. (2) If $K(p) < 0$ there are two asymptotic directions at p bisected by the principal directions at an angle ϑ given by $\tan^2\vartheta = -k_1(p)/k_2(p)$. (3) If $K(p) = 0$ and p is a planar point every direction at p is asymptotic and principal; otherwise there is only one asymptotic direction at p, which is also principal.	Page 225, Lemma 5.4.
(vi)	A curve α is asymptotic if and only if $S(\alpha')\cdot\alpha' = -U'\cdot\alpha' = U\cdot\alpha'' = 0$.	Page 226, line -3.

(vii)	The curves given by the intersection of a surface M with the tangent plane $\overline{T}_p(M)$ point in asymptotic directions at p.	Text, page 41
(viii)	A surface is minimal if and only if there exist two orthogonal asymptotic directions at each of its points.	Page 227, line 4
(ix)	A ruled surface has Gaussian curvature $K \leqslant 0$. $K = 0$ if and only if the unit normal U is parallel along each of the rulings.	Page 227, Lemma 5.6
(x)	A straight line in a surface is asymptotic.	Page 227, line -4
(xi)	A geodesic has constant speed.	Page 228, line 18
(xii)	A straight line in a surface is a geodesic.	Page 228, line 21
(xiii)	For unit-speed geodesics, $S(T) = \kappa T - \tau B$.	Page 228, line -11.

Techniques

(i) Using Results (i) – (iv) and (vi) to recognize or find principal curves or asymptotic curves.

(ii) Using the definition to recognize a geodesic.

Exercises

Technique (i)

1. Page 230, Exercise 2. Apply as many as possible of the techniques based on Results (i) – (iv) and (vi) to the three curves.

Technique (ii)

2. Are the curves of Exercise 1 geodesics?

Theory Exercise

3. Page 230, Exercise 7. This exercise gives an approach to curves on a surface that encompasses principal curves, asymptotic curves and geodesics.

Solutions

1. Page 230, Exercise 2.

(a) The top circle α on the torus.

The standard parametrization of the torus is

$$x(u, v) = ((R + r\cos u)\cos v,\ (R + r\cos u)\sin v,\ r\sin u).$$

The top circle is the v-parameter curve $u = \frac{\pi}{2}$. This is given by

$$\alpha : t \longmapsto x(\tfrac{\pi}{2}, t) = (R\cos t,\ R\sin t,\ r).$$

By symmetry arguments the unit vector field normal to the surface along this curve points in the direction (0, 0, 1). Hence on the curve the normal vector field is $U:t \longmapsto (0, 0, 1)_{\alpha(t)}$.

To apply the techniques we need to compute the following:

$$\alpha' : t \longmapsto (-R \sin t,\ R \cos t,\ 0)$$
$$\alpha'' : t \longmapsto (-R \cos t,\ R \sin t,\ 0)$$
$$U' : t \longmapsto (0, 0, 0)_{\alpha(t)}$$
$$S(\alpha') = -U' = (0, 0, 0).$$

Now we can apply Results (i) – (iv), (vi).

(i) Since $U' = \mathbf{0}$ it follows that $S(\alpha') = 0\alpha'$ and hence α is principal.

(ii) The principal curvature of α is 0.

(iii) α is the intersection of the plane $z = r$ with the torus, and the normal to that plane is equal to the normal to the surface along α. Hence again α is a principal curve.

(iv) α is a parallel and hence is a principal curve.

(vi) Since $S(\alpha') = \mathbf{0}$ it follows that $S(\alpha')\cdot\alpha' = 0$ and hence α is asymptotic.

(b) The outer equator β of the torus.

Using the same parametrization as in (a) the outer equator is given by

$$\beta : t \longmapsto x(0, t) = ((R + r)\cos t,\ (R + r)\sin t,\ 0).$$

By symmetry arguments the unit vector field normal to the surface along this curve points radially outwards (or alternatively inwards) and hence is given by

$$U : t \longmapsto (\cos t,\ \sin t,\ 0)_{\beta(t)}.$$

We calculate

$$\beta' : t \longmapsto (-(R + r)\sin t,\ (R + r)\cos t,\ 0)$$
$$\beta'' : t \longmapsto (-(R + r)\cos t,\ -(R + r)\sin t,\ 0)$$

and $U' : t \longmapsto (-\sin t,\ \cos t,\ 0)$.

Now we can apply Results (i) – (iv), (vi).

(i) Since $S(\beta') = -U' = -\dfrac{\beta'}{(R + r)}$ the curve β is principal.

(ii) The principal curvature along β is the constant function $-1/(R + r)$.

(iii) β is the intersection of the torus with part of the plane $z = 0$. The normal to this plane is vertical while the normal to the torus along β is always horizontal. Therefore they make a constant angle of $\frac{\pi}{2}$ with each other and so β is a principal curve.

(iv) β is a parallel and hence a principal curve.

(vi) Since $\beta''\cdot U = -(R + r) \neq 0$ we see that β is not an asymptotic curve.

(c) The x axis on $M: z = xy$.

The x axis is the curve

$$\gamma : t \longmapsto (t, 0, 0).$$

Since M is defined implicitly as M:g = 0, where $g = z - xy$, we can use ∇g to obtain a unit vector field normal to M. This is

$$U = \frac{(-y, -x, 1)}{(1 + x^2 + y^2)^{\frac{1}{2}}}.$$

Its restriction to γ is

$$U : t \longmapsto \frac{(0, -t, 1)}{(1 + t^2)^{\frac{1}{2}}}.$$

Next we find:

$$\gamma' : t \longmapsto (1, 0, 0)$$

$$\gamma'' : t \longmapsto (0, 0, 0)$$

$$U' : t \longmapsto \frac{(0, -1, -t)}{(1 + t^2)^{\frac{3}{2}}}.$$

Now we can apply Results (i) and (vi).

(i) Since U' and γ' are not collinear, γ is not a principal curve.

(vi) Since $\gamma'' = \mathbf{0}$ it follows that $U \cdot \gamma'' = 0$ and hence γ is an asymptotic curve.

2. Page 230, Exercise 2.

(a) Since $\alpha'' \neq \mathbf{0}$ while $\alpha'' \cdot U = 0$ the acceleration is not normal to the surface and hence α is not a geodesic.

(b) Since $\beta'' = -(R + r)U$ the curve β is a geodesic.

(c) Since $\gamma'' = \mathbf{0} = 0U$ the curve γ is a geodesic.

3. Page 230, Exercise 7.

(a) Since T, V, U form a frame field orthonormal expansion gives

$$T' = (T' \cdot T)T + (T' \cdot V)V + (T' \cdot U)U$$

$$V' = (V' \cdot T)T + (V' \cdot V)V + (V' \cdot U)U$$

$$U' = (U' \cdot T)T + (U' \cdot V)V + (U' \cdot U)U.$$

Differentiating the equations

$$T \cdot T = V \cdot V = U \cdot U = 1$$

and

$$T \cdot V = V \cdot U = U \cdot T = 0$$

we obtain

$$T' \cdot T = V' \cdot V = U' \cdot U = 0$$

and

$$T' \cdot V + V' \cdot T = V' \cdot U + U' \cdot V = U' \cdot T + T' \cdot U = 0.$$

Setting $g = T' \cdot V$, $k = T' \cdot U$ and $t = V' \cdot U$ the result follows.

Since $T = \alpha'$ it follows that

$$k = T' \cdot U = \alpha'' \cdot U = S(\alpha') \cdot \alpha' = S(T) \cdot T.$$

Since $t = V' \cdot U$ it follows that

$$t = V' \cdot U = -U' \cdot V = S(\alpha') \cdot V = S(T) \cdot V.$$

(b) α is geodesic $\iff \alpha''$ is collinear to U

$\iff T'$ is collinear to U

$\iff g = 0.$

α is asymptotic $\iff S(\alpha') \cdot \alpha' = 0$

$\iff S(T) \cdot T = 0$

$\iff k = 0.$

α is principal $\iff \alpha'$ and $S(\alpha')$ are collinear

$\iff T$ and $-U'$ are collinear

$\iff t = 0.$

V.6 SURFACES OF REVOLUTION

Introduction

This section follows on from Sections IV.2 and V.4.

In Example IV.2.5 we introduced a way of parametrizating surfaces of revolution and in Section V.4 we obtained formulas for the Gaussian and mean curvature of surfaces given by a parametrization. In this section we tie these two sections together and work out explicit formulas for curvature of surfaces of revolution.

We then describe surfaces of revolution that are minimal or have constant Gaussian curvature.

READ: Section V.6 (pages 234-242) omitting cases 1, 2 and 3 on pages 240-241.

Comment

(i) *Page 235: first three lines* Since S maps $T_p(M)$ linearly to itself and $\{x_u, x_v\}$ is a basis for $T_p(M)$ we must have

$$S(x_u) = ax_u + bx_v$$

$$S(x_v) = cx_u + dx_v$$

for some functions a, b, c, d. Forming dot products with x_u and x_v we obtain the equations

$$\ell = aE + bF \qquad m = aF + bG$$

$$m = cE + dF \qquad n = cF + dG.$$

Since $F = m = 0$ it follows that

$$b = c = 0 \text{ and } a = \ell/E,\ d = n/G.$$

Hence $S(x_u) = (\ell/E)x_u$ and $S(x_v) = (n/G)x_v$.

Supplementary Comment

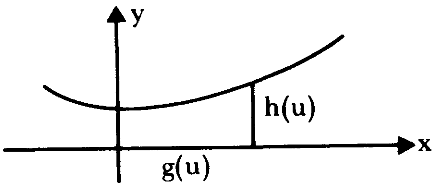
(i) *Page 236: Theorem 6.2*

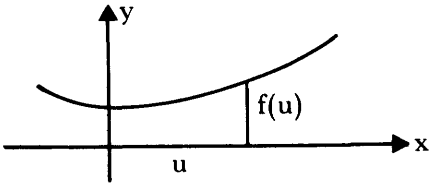
Case 1: If $g = c$ (a constant) then

$$x(u, v) = (c, h(u) \cos v,\ h(u) \sin v)$$

and M belongs to the plane $x = c$. This is a plane parallel to the yz plane and it passes through the point $(c, 0, 0)$.

Case 2: If g' is never zero then either g is strictly increasing or strictly decreasing. In either case the profile curve passes over each point of the x axis once only and so we can reparametrize it so that it is the graph of a function f. That is, we can replace g and h by the functions u and f.





Then by the formulas on page 235

$$2H = k_\pi + k_\mu = \frac{1}{f(1+f'^2)^{\frac{1}{2}}} - \frac{f''}{(1+f'^2)^{\frac{3}{2}}}.$$

Since M is minimal $H = 0$, and hence multiplying through by $f(1+f'^2)^{\frac{3}{2}}$ we obtain

$$1 + f'^2 = ff''.$$

To solve this equation we first differentiate it to obtain

$$2f'f'' = ff''' + f'f''.$$

Hence $f'f'' = ff'''$;

$$\frac{f'}{f} = \frac{f'''}{f''}.$$

Integrating we find

$$\log f = \log f'' - \log c$$

and hence $f'' = cf$ for some constant c.

Since the graph of f does not intersect the axis, we can assume that f is positive and hence so is $f'' = (1 + f'^2)/f$. Hence $c > 0$ and we can write $c = 1/a^2$. Then we can solve

$$f'' = \frac{1}{a^2} f$$

to obtain

$$f(u) = d \cosh\left(\frac{u}{a} + b\right),$$

where d and b are constants. Substituting in the original equation we find that $d = a$ and hence the solution is as given.

Summary

Definitions

(i)	*Standard* parametrization of a surface of revolution: $x(u, v) = (g(u), h(u) \cos v,\ h(u) \sin v)$	Page 234, line 12
(ii)	*Special* parametrization of a surface of revolution passing at most once over each point of the axis: $x(u, v) = (u, h(u) \cos v,\ h(u) \sin v)$	Page 235, line -10
(iii)	*Canonical* parametrization of a surface of revolution: as for the standard parametrization but with a unit speed profile curve, i.e. $g'^2 + h'^2 = 1$	Page 238, line 12

Examples

(i)	Torus of revolution	Page 235, Example 6.1(1)
(ii)	Catenoid	Page 236, Example 6.1(2)
(iii)	Bugle surface	Page 241, Example 6.6

Results

(i) For the standard parametrization

$$k_\mu = -\frac{\begin{vmatrix} g' & h' \\ g'' & h'' \end{vmatrix}}{(g'^2 + h'^2)^{\frac{3}{2}}} \qquad k_\pi = \frac{g'}{h(g'^2 + h'^2)^{\frac{1}{2}}}$$

and $K = -\dfrac{g'\begin{vmatrix} g' & h' \\ g'' & h'' \end{vmatrix}}{h(g'^2 + h'^2)^2}$.

Page 235, lines 7 and 9

(ii) For the special parametrization

$$k_\mu = \frac{-h''}{(1 + h'^2)^{\frac{3}{2}}}, \qquad k_\pi = \frac{1}{h(1 + h'^2)^{\frac{1}{2}}}$$

and $K = \dfrac{-h''}{h(1 + h'^2)^2}$.

Page 235, lines -6 to -4

(iii) If a surface of revolution is minimal then it is contained in either a plane or a catenoid.

Page 236, Theorem 6.2

(iv) For the canonical parametrization

$$K = -h''/h.$$

Page 238, Lemma 6.3

(v) A surface of revolution with curvature K is obtained as follows. First find a function h such that $h'' + Kh = 0$, with $h(0) > 0$ and $|h'(0)| < 1$ and then find a function g such that

$$g(u) = \int_0^u (1 - h'(t)^2)^{\frac{1}{2}}\, dt.$$

Then the profile curve for the surface is $\alpha(u) = (g(u), h(u), 0)$.

Page 239, lines -13 to -2

(vi) For a torus of revolution

$$K = \frac{\cos u}{r(R + r\cos u)}.$$

Page 235, Example 6.1(1)

(vii) For a catenoid

$$K = \frac{-1}{c^2 \cosh^4 (u/c)}.$$

Page 236, Example 6.1(2)

(viii) A canonical parametrization of the catenoid is given by the unit speed profile curve

$$\beta(s) = (\sinh^{-1} s, (1 + s^2)^{\frac{1}{2}})$$

and for this parametrization $K(s) = \dfrac{-1}{(1 + s^2)^2}$.

Page 238, Example 6.4

(ix) The bugle surface has constant curvature $-1/c^2$.

Techniques

(i)	Calculating the principal and Gaussian curvature for surfaces of revolution using the formulas for standard, special and canonical parametrizations.	Page 235, lines 7 and 9 and Page 235, lines −6 to −4 and Page 238, Lemma 6.3.
(ii)	Finding surfaces of revolution with prescribed Gaussian curvature.	Page 239, lines −13 to −2.

Exercises

Technique (i)

1. Let α be the profile curve that is the top half of the ellipse $x^2/a^2 + y^2/b^2 = 1$. We can parametrize this as follows:

$$\alpha(u) = (a \cos u,\ b \sin u) \qquad \text{where } 0 < u < \pi.$$

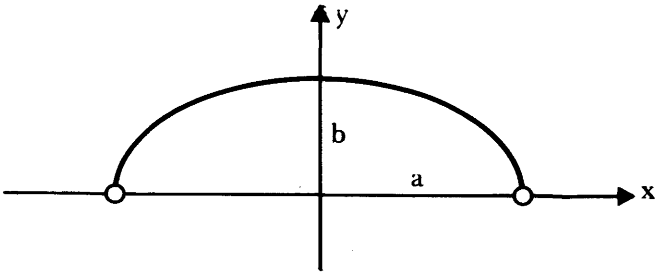
Find a special parametrization and use both this and the given parametrization to calculate the Gaussian curvature of the surface of revolution obtained by revolving about the x axis. This surface is an ellipsoid.

Technique (ii)

2. Find those surfaces of revolution with zero Gaussian curvature. Use the technique described in Result (v) and also consider the case when $|h'(0)| = 1$.

Solutions

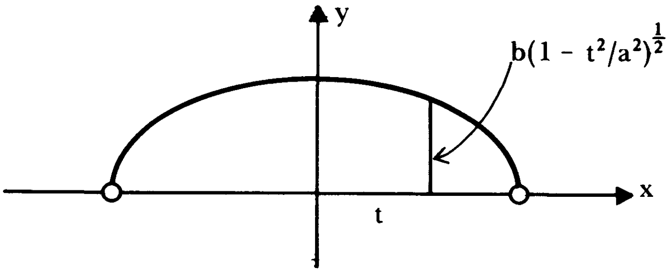
1. The ellipse appears as follows.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

A special parametrization is given by the profile curve

$$\beta(t) = (t,\ b(1 - t^2/a^2)^{\frac{1}{2}}), \qquad \text{where } a > t > -a.$$



We have been given a standard parametrization with $g(u) = a\cos u$ and $h(u) = b\sin u$ and hence using the formula (2) on page 235

$$K = -\frac{g'\begin{vmatrix} g' & h' \\ g'' & h'' \end{vmatrix}}{h(g'^2 + h'^2)^2}$$

so $$K(u) = \frac{a\sin u\begin{vmatrix} -a\sin u & b\cos u \\ -a\cos u & -b\sin u \end{vmatrix}}{b\sin u\,(a^2\sin^2 u + b^2\cos^2 u)^2}$$

$$= \frac{a}{b}\,\frac{(ab\sin^2 u + ab\cos^2 u)}{(a^2\sin^2 u + b^2\cos^2 u)^2} = \frac{a^2}{(a^2\sin^2 u + b^2\cos^2 u)^2}.$$

For the special parametrization

$$h(t) = b(1 - t^2/a^2)^{\frac{1}{2}} = \frac{b}{a}(a^2 - t^2)^{\frac{1}{2}}$$

$$h'(t) = -bt/a(a^2 - t^2)^{\frac{1}{2}}, \quad h''(t) = -ba/(a^2 - t^2)^{\frac{3}{2}}$$

$$1 + h'(t)^2 = \frac{a^4 + (b^2 - a^2)t^2}{a^2(a^2 - t^2)}.$$

Hence $$K(t) = \frac{-h''(t)}{h(t)(1 + h'(t)^2)^2} = \frac{a^6}{(a^4 + (b^2 - a^2)t^2)^2}.$$

If we evaluate the second formula for K at the point $t = a\cos u$ we should obtain the first formula.

$$K(a\cos u) = \frac{a^6}{(a^4 + (b^2 - a^2)a^2\cos^2 u)^2}$$

$$= \frac{a^2}{(a^2(1 - \cos^2 u) + b^2\cos^2 u)^2}$$

$$= \frac{a^2}{(a^2\sin^2 u + b^2\cos^2 u)^2}, \qquad \text{as we expected.}$$

2. Since $K = 0$ we need to solve

$$h'' + 0h = h'' = 0$$

with $h(0) > 0$ and $|h'(0)| \leqslant 1$.

We obtain $h(t) = at + b$ with $|a| \leqslant 1$. We next find the function

$$g(u) = \int_0^u (1 - h'(t)^2)^{\frac{1}{2}}dt = (1 - a^2)^{\frac{1}{2}}u.$$

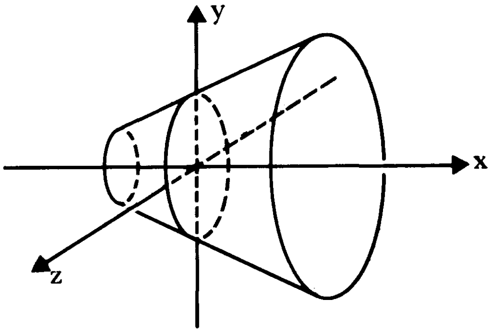
Hence the surface of revolution is obtained by revolving the curve

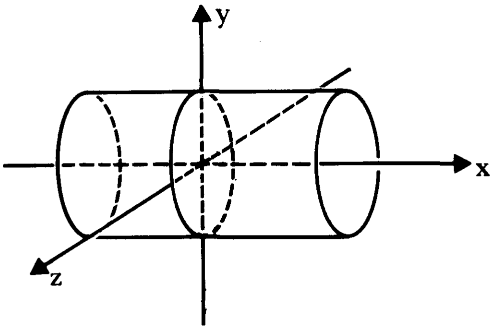
$$t \longmapsto ((1 - a^2)^{\frac{1}{2}}t, at + b)$$

about the x axis. Now, if $1 - a^2 \neq 0$ (that is, $|h'(0)| < 1$) we can reparametrize the profile curve so that $g(t) = t$. Hence we see that the surface can be obtained by revolving the straight line

$$t \longmapsto (t, ct + b).$$

The surfaces we obtain are parts of cylinders (when $a = c = 0$) and parts of cones.





On the other hand, if $1 - a^2 = 0$ (that is, $|h'(0)| = 1$) then $g' = 0$ and so g is identically zero. Thus the surface is part of the yz plane.

FURTHER EXERCISES AND SOLUTIONS

Section V.1

Recommended Exercise

Page 195, Exercise 9. This continues Exercise II.5.7. You will have to assume that W is also a tangent vector field on M.

Solution

9.
$$
\begin{aligned}
& U\cdot W = 0 \\
\Rightarrow\ & \nabla_V U\cdot W + \nabla_V W\cdot U = 0 \\
\Rightarrow\ & S(V)\cdot W = \nabla_V W\cdot U. \\
& S(V)\cdot W = S(W)\cdot V \\
\Longleftrightarrow\ & \nabla_V W\cdot U - \nabla_W V\cdot U = 0 \\
\Longleftrightarrow\ & [V, W]\cdot U = 0.
\end{aligned}
$$

Section V.2

Technique Exercise

Page 202, Exercise 3.

Solution

$$
\begin{aligned}
p &= \alpha_n(0) = (r, 0, 0) \\
U(p) &= (1, 0, 0) \\
\alpha_n''(t) &= (-r\cos t,\ -r\sin t,\ \pm n(n-1)t^{n-2}) \\
\alpha_n''(0) &= (-r, 0, 0) \quad \text{if} \quad n > 2; \qquad \alpha_2''(0) = (-r, 0, \pm 2) \\
\alpha_n''(0)\cdot U(p) &= -r.
\end{aligned}
$$

Section V.3

Technique Exercises

Pages 207-8, Exercises 2, 5(a), (b). The final zero in 2(b) and 2(c) should be in bold type. (HINT for Exercise 2: Express each condition in terms of the matrix for S_p with respect to $\{\mathbf{u}_1, \mathbf{u}_2\}$.)

Other Recommended Exercise

Page 208, Exercise 4(a). This gives an alternative proof of Corollary V.3.5 by finding the equation whose roots are the principal curvatures.

Solutions

2(a). $\mathbf{u}_1$ and $\mathbf{u}_2$ are principal vectors.

2(b). $(\mathbf{u}_1 \pm \mathbf{u}_2)/\sqrt{2}$ are principal vectors and $K(p) = 0$.

2(c). $K(p) = 0$

2(d). $\mathbf{u}_1$ and $\mathbf{u}_2$ are principal vectors or $H(p) = 0$.

5(a). A single point; two intersecting straight lines; one straight line.

5(b). An ellipse, the empty set; hyperbola, hyperbola; two parallel straight lines, the empty set.

4(a). If T is any 2-dimensional linear operator, with matrix A, then $\det(A - kI) = k^2 - (\text{Trace } A)\, k + \det A$, as may be verified by evaluating each side.

Section V.4

Technique Exercises

Pages 219-221, Exercises 1, 17.

Other Recommended Exercises

Page 220, Exercises 5 and 6. These extend the exercises on curvature of a Monge patch.

Page 220, Exercise 9. This deals with the matrix representation of S in simple cases.

Pages 220-221, Exercises 10, 11. These deal with a coordinate description of principal vectors.

Solutions

1. $\ell = -r\cos^2 v \qquad W = r^2 \cos v$

$m = 0 \qquad K = 1/r^2$

$n = -r$

$$Z = \tfrac{1}{2}\nabla g = \frac{x}{a^2}U_1 + \frac{y}{b^2}U_2 - \frac{z}{c^2}U_3$$

$$\nabla_V Z = \frac{v_1}{a^2}U_1 + \frac{v_2}{b^2}U_2 - \frac{v_3}{c^2}U_3$$

$$Z\cdot\nabla_V Z \times \nabla_W Z = \frac{1}{a^2b^2c^2}\begin{vmatrix} x & y & -z \\ v_1 & v_2 & -v_3 \\ w_1 & w_2 & -w_3 \end{vmatrix}$$

$$= \frac{-1}{a^2b^2c^2}\, X\cdot V \times W = -\frac{1}{a^2b^2c^2}\, X\cdot Z$$

$$= \mp \frac{1}{a^2b^2c^2}$$

$$K = \mp \frac{1}{a^2b^2c^2\,\|Z\|^4}$$

$$h = X\cdot U = \frac{X\cdot Z}{\|Z\|} = \pm\frac{1}{\|Z\|}$$

$$K = \mp h^4/a^2b^2c^2$$

5. $K = -36r^2/(1+9r^4)^2$

No

6. $K = 4\varepsilon/a^2b^2\left(1 + 4\frac{u^2}{a^4} + 4\frac{v^2}{b^4}\right)^2$

9(a). $S(x_u) = ax_u + bx_v.$

$S(x_v) = cx_u + dx_v.$

Hence $S(x_u)\cdot x_u = ax_u\cdot x_u$, implying $a = \ell/E$, and so on.

9(b). $F = m = 0$ implies

$$S(x_u) = (\ell/E)x_u \text{ and } S(x_v) = (n/G)x_v.$$

10. $$x_u \times x_v\cdot(S(v) \times v) = \begin{vmatrix} x_u\cdot S(v) & x_u\cdot v \\ x_v\cdot S(v) & x_v\cdot v \end{vmatrix}$$

$$= \begin{vmatrix} v_1\ell + v_2 m & v_1E + v_2F \\ v_1 m + v_2 n & v_1F + v_2G \end{vmatrix}$$

$$= v_1^2(\ell F - mE) + v_1v_2(\ell G - nE) + v_2^2(mG - nF)$$

$$= -\begin{vmatrix} v_2^2 & -v_1v_2 & v_1^2 \\ E & F & G \\ \ell & m & n \end{vmatrix}$$

11. $V = (1+u^2)^{\frac{1}{2}}x_u \pm (1+v^2)^{\frac{1}{2}}x_v$

$$\begin{vmatrix} 1+v^2 & \mp((1+u^2)(1+v^2))^{\frac{1}{2}} & 1+u^2 \\ 1+v^2 & uv & 1+u^2 \\ 0 & 1/W & 0 \end{vmatrix} = 0$$

Section V.5

Technique Exercises

Pages 229-232, Exercises 1, 3, 4, 5, 10, 11, 12.

Other Recommended Exercises

Page 230, Exercise 6. This deals with a coordinate description of principal and asymptotic curves.

Page 231, Exercise 8. This extends Exercise V.5.7.

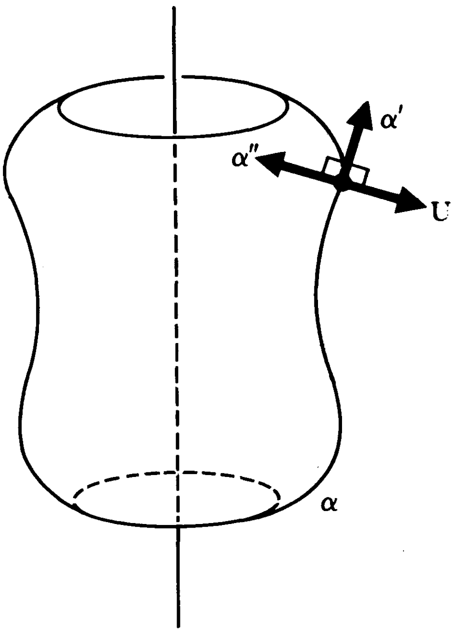
Solutions

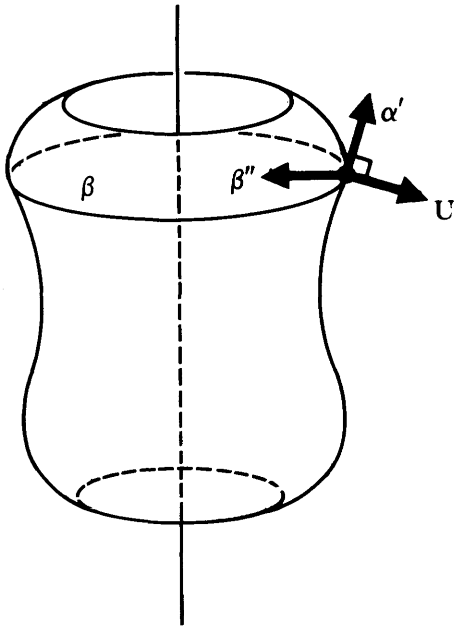
1. α is a straight line $\Rightarrow \alpha'' = 0$

$\Rightarrow \alpha$ is geodesic and asymptotic.

α is geodesic and asymptotic $\Rightarrow \alpha'' \cdot U = 0$ and α'' and U are collinear $\Rightarrow \alpha'' = 0$ $\Rightarrow \alpha$ is a straight line.

3.





4. Assume α is unit-speed.

(a) α is asymptotic $\Rightarrow 0 = \alpha'' \cdot U = T' \cdot U = \kappa N \cdot U$. Hence T and N are tangents to the surface and B is normal.

So $S(T) = -B' = \tau N$.

(b) Since S is symmetric,

$$S(N) = \tau T + \lambda N$$

for some λ. The matrix for S is $\begin{pmatrix} 0 & \tau \\ \tau & \lambda \end{pmatrix}$

$$K = \det S = -\tau^2.$$

(c) Since $n = 0$ the v-parameter curves $u = u_0$ are asymptotic. A typical one of these is the helix

$$\alpha : v \longmapsto (u_0 \cos v,\ u_0 \sin v,\ bv).$$

$$\tau = b/(b^2 + u_0{}^2)$$

$$K = -b^2/(b^2 + u_0{}^2)^2.$$

5. α is principal in M $\Rightarrow k\alpha' = -U'$.

But $U \cdot \bar{U}$ = constant $\Rightarrow U' \cdot \bar{U} + U \cdot \bar{U}' = 0$.

Hence $0 = -k\alpha' \cdot \bar{U} + U \cdot \bar{U}' = U \cdot \bar{U}'$.

Thus $\bar{U}'$ and α' are both orthogonal to U and to $\bar{U}$: if these are linearly independent then α' and $\bar{U}'$ are collinear and so α is principal in $\bar{M}$.

If U and $\bar{U}$ are linearly dependent then $U = \pm \bar{U}$ and so α is principal in M if and only if α is principal in $\bar{M}$.

10. $x_u \times x_v = \beta' \times \delta + v\delta' \times \delta$

$n = 0$

$m = x_{uv} \cdot (x_u \times x_v)/W = \delta' \cdot \beta' \times \delta/W = \beta' \cdot \delta \times \delta'/W$

11(a). Cone: $x = p + v\delta$.

$\beta = p; \beta' = 0; K = 0.$

Cylinder: $x = \beta + vq$.

$\delta = q; \delta' = 0; K = 0.$

11(b). $K = -(\beta' \cdot T \times T')^2/W^4$

$= 0, \qquad \text{since } \beta' = T.$

12. $K = -(\alpha' \cdot U \times U')/W^4$

$= -(U \cdot U' \times \alpha')/W^4$

$K = 0 \Longleftrightarrow U' \times \alpha' = 0$

$\Longleftrightarrow \alpha'$ and U' are collinear

$\Longleftrightarrow \alpha$ is principal.

6(a). $\alpha' = a_1' x_u + a_2' x_v$. Use Exercise V.4.10.

6(b). $S(\alpha') \cdot \alpha' = \ell a_1'^2 + 2m a_1' a_2' + n a_2'^2$.

8(a). α is principal and geodesic

$\Longleftrightarrow g = t = 0$

$\Longleftrightarrow V =$ constant

$\Longleftrightarrow \alpha$ lies in a plane orthogonal to M along α.

8(b). α is principal and asymptotic

$\Longleftrightarrow t = k = 0$

$\Longleftrightarrow U =$ constant

$\Longleftrightarrow \alpha$ lies in a plane tangent to M along α.

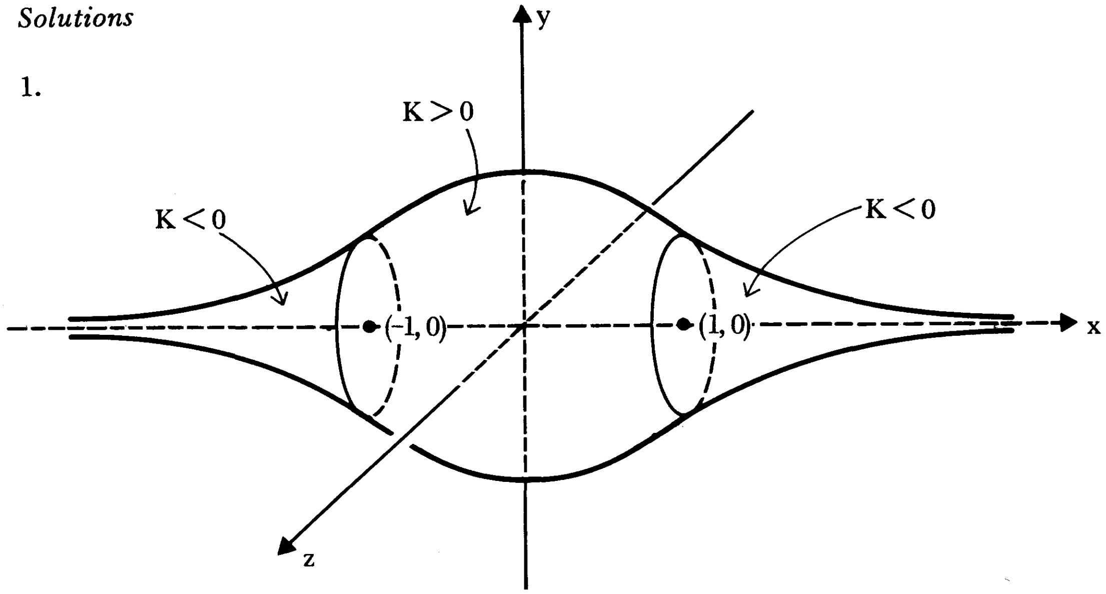
Section V.6

Technique Exercises

Pages 242-243, Exercises 1, 3, 4, 5, 6, 8. (Errata in Exercise 3: in line -3 "h(u)" should be "$h(u)^{-1}$", and in line -1 "h" should be "h^{-1}".

Solutions

1.



$g(u) = u$, $h(u) = \exp(-u^2/2)$,

$K(u) = (1 - u^2)/(1 + u^2 e^{-u^2})^2$.

3. $\alpha = (g, h, 0)$

$\alpha' = (g', h', 0)$

$\alpha'' = (g'', h'', 0)$

$\kappa = \|\alpha' \times \alpha''\| / \|\alpha'\|^3 = |g'h'' - h'g''| / (g'^2 + h'^2)^{\frac{3}{2}}$

$= |k_\mu|$

$|\cos\varphi| = \dfrac{|g'|}{(g'^2 + h'^2)^{\frac{1}{2}}} = h|k_\pi|$

$|K| = |k_\mu|\,|k_\pi| = \dfrac{\kappa}{h}|\cos\varphi|$

4. $\alpha(u) = (R + a\cos u,\ b\sin u,\ 0)$

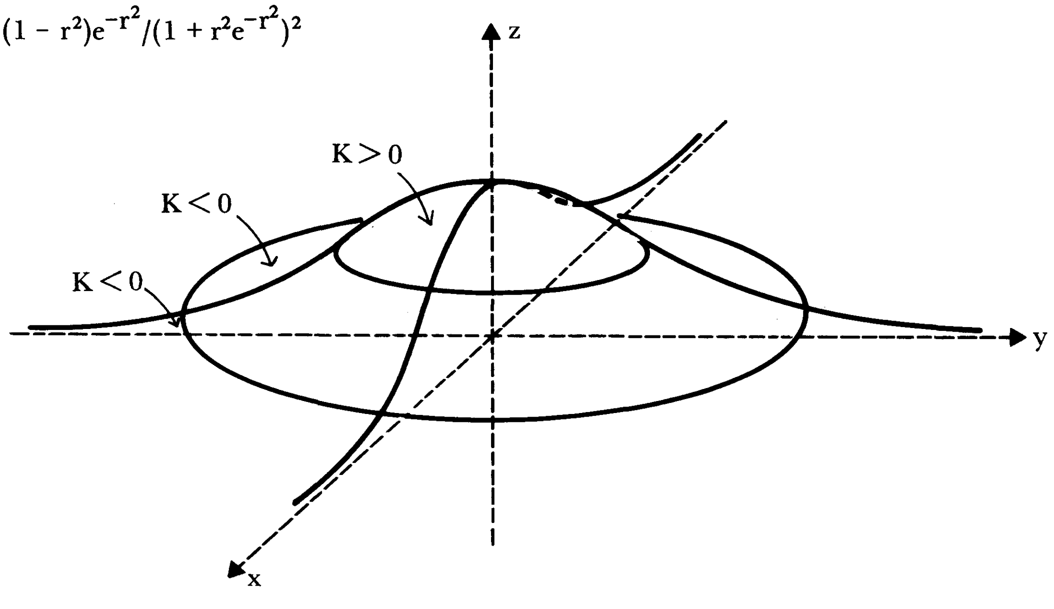
$K(u) = b^2 a\cos u/(R + a\cos u)(b^2\cos^2 u + a^2\sin^2 u)^2$

5. $x(r, v) = (r\cos v,\ r\sin v,\ f(r))$

$g(r) = f(r)$, $h(r) = r$,

$K(r) = f'(r)f''(r)/r(1 + f'(r)^2)^2$.

6. $K(r) = (1 - r^2)e^{-r^2}/(1 + r^2 e^{-r^2})^2$



8(a). $g(u) = u + \sin u$, $h(u) = 1 + \cos u$,

$K(u) = 1/4(1 + \cos u)$.

8(b). $s(t) = 4\sin(t/2)$

$h(s) = 2 - s^2/8$

$\overline{K}(s) = 1/4(2 - s^2/8)$.

8(c). $1/4(2 - s^2/8) = 1/4(1 + \cos t)$.

DIFFERENTIAL GEOMETRY

I Calculus on Euclidean Space

II Frame Fields

III Euclidean Geometry

IV Calculus on a Surface

V Shape Operators

VI Geometry of Surfaces in E^3